Physics PhD Qualifying Examination
Part I – Wednesday, January 21, 2015

Name: **(problems & solutions)**
(please print)
Identification Number:________

STUDENT: Designate the problem numbers that you are handing in for grading in the appropriate left hand boxes below. Initial the right hand box.
PROCTOR: Check off the right hand boxes corresponding to the problems received from each student. Initial in the right hand box.

	1	
	2	
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	8	
	9	
	10	

Student's initials
problems handed in:
Proctor's initials

INSTRUCTIONS FOR SUBMITTING ANSWER SHEETS

1. DO NOT PUT YOUR NAME ON ANY ANSWER SHEET. EXAMS WILL BE COLLATED AND GRADED BY THE ID NUMBER ABOVE.
2. Use at least one separate preprinted answer sheet for each problem. Write on only one side of each answer sheet.
3. Write your **identification number** listed above, in the appropriate box on each preprinted answer sheet.
4. Write the **problem number** in the appropriate box of each preprinted answer sheet. If you use more than one page for an answer, then number the answer sheets with both problem number and page (e.g. Problem 9 – Page 1 of 3).
5. Staple together all the pages pertaining to a given problem. Use a paper clip to group together all problems that you are handing in.
6. A passing distribution for the individual components will normally include at least three passed problems (from problems 1-5) for Mechanics and three problems (from problems 6-10) for Electricity and Magnetism.
7. **YOU MUST SHOW ALL YOUR WORK.**

I-1 [10]

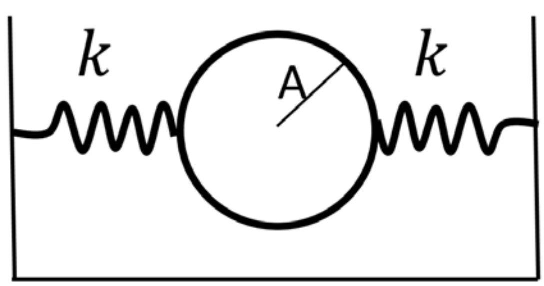
A ball of mass m is thrown with initial speed v from the edge of a cliff of height h.

- **(a)** At what inclination angle θ_{max} should the ball be thrown so that it travels the maximum horizontal distance d_{max}?
- **(b)** What is the maximum horizontal distance d_{max}?

Assume that the ground below the cliff is horizontal.

I-2 [10]

A thin disk of mass M and radius A is connected by two springs of spring constant k to two fixed points on a frictionless table top. The disk is free to rotate but its center is constrained to move in one dimension within the plane. Each spring has an unstretched length of l_0, and initially both are stretched to a length $l > l_0$ in the equilibrium position, as shown in the figure below. What are the frequencies of the normal modes of oscillation for small vibrations? Indicate the nature of motion for each mode.



I-3 [10]

A *nonuniform* wire of total length L and total mass M has a *variable linear mass density* given by $\mu(x) = kx$, where x is the distance measured from one end of the wire and k is a constant. The tension in the wire is F.

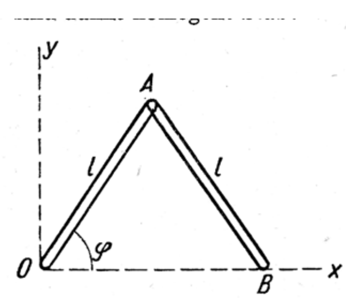
(a) Determine k in terms of M and L.

(b) How long does it take for a transverse pulse generated at one end of the wire to travel to the other end? You must express your answer in terms of M, L, and F.

I-4 [10]

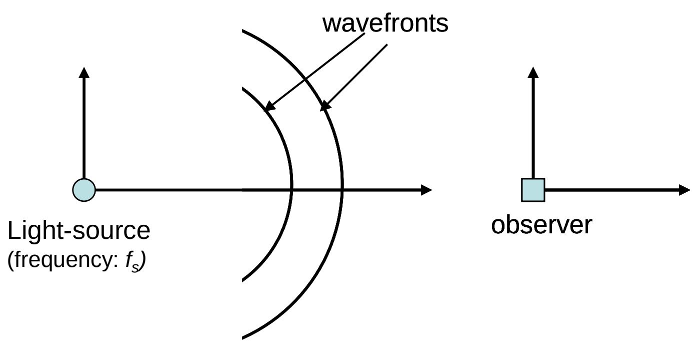
Consider a system consisting of two thin long rods of equal uniform masses *m*, equal lengths *l* and equal radii *R*. The two rods are connected by a (massless) hinge in point A. Rod OA rotates around O. The rotation is in the *x*-*y*-plane. The end B of rod AB slides without friction along the x-axis as illustrated in the figure below.

Obtain the total kinetic energy of the system as a function of φ, $\dot{\varphi}$, and the parameters given above.



I-5 [10]

The ***relativistic Doppler effect*** is the change in frequency f of light, caused by relative motion of the source and the observer. Assume that the source and the observer are moving ***away*** from each other with a relative velocity v.



Consider the problem in the ***reference frame of the source***. Let f_s be the frequency of the wave the source emitted. Suppose one wavefront arrives at the observer.

(a) What is the distance of the next wavefront away from him?

(b) What is the time t between crest (of the wavefront) arrivals at the observer?

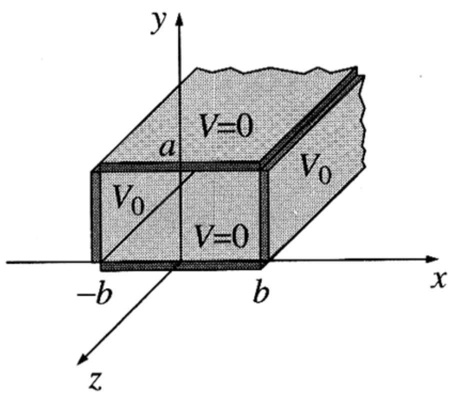
(c) Due to relativistic effect, what will the observer measure this time t_0 to be?

(d) What is the corresponding ***observed frequency*** f_0?

I-6 [10]

Two infinitely long grounded metal plates at $y=0$ and $y=a$ are connected at $x=\pm b$ by metal strips maintained at a constant electrical potential V_o as shown in the figure. A thin layer of insulation at each corner prevents the plates from shorting out.

Calculate the potential $\phi(x,y,z)$ inside the pipe.



I-7 [10]

(a) Write down Maxwell's equations for free space where there are no current or charge distributions.

(b) Derive wave equations for electric and magnetic fields from Maxwell's equations.

(c) Assume plane-wave solution for the fields and show that ***E, B*** and ***k*** are perpendicular to each other.

$$\vec{E} = \vec{E_0}\cos(\vec{k} \bullet \vec{r} - \omega t)$$
$$\vec{B} = \vec{B_0}\cos(\vec{k} \bullet \vec{r} - \omega t)$$
$$c = \omega / k$$

(d) Use the two curl equations in Maxwell's equations to show the relationship between ε_0, μ_0 and speed-of-light in free space, c.

I-8 [10]

A long, solid dielectric cylinder of radius a is permanently polarized so that the polarization is everywhere radially outward, with a magnitude proportional to the distance from the axis of the cylinder, i.e., $\boldsymbol{P} = \frac{1}{2} P_0 r \hat{r}$.

(a) Find the charge density in the cylinder.

(b) If the cylinder is rotated with a constant angular velocity ω about its axis without change in $\boldsymbol{P}$, what is the magnetic field on the axis of the cylinder at point not too near its ends?

I-9 [10]

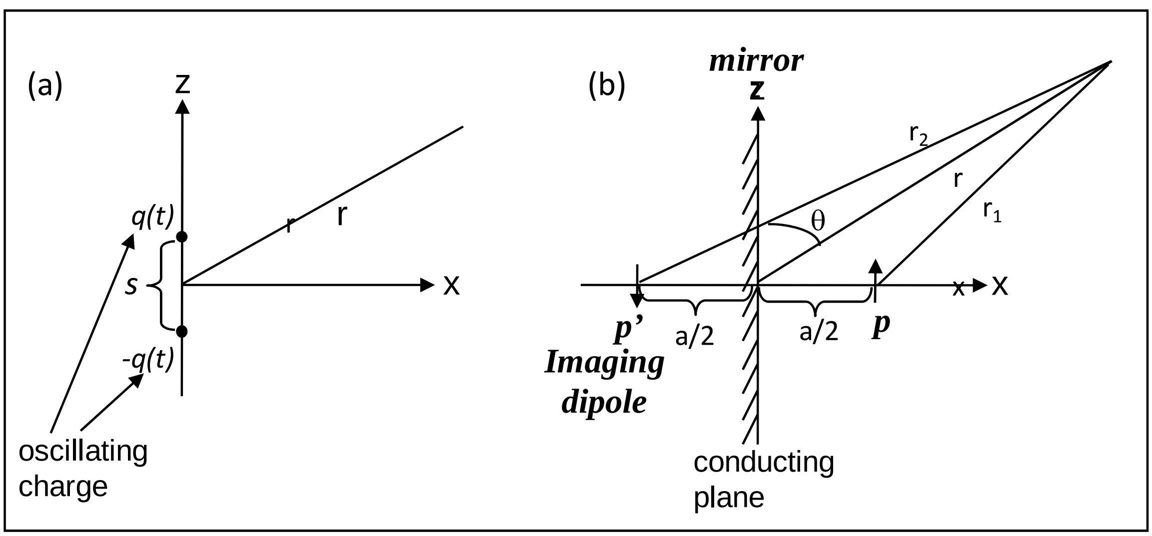
A system of two tiny metal spheres separated by a distance s and connected by a fine wire is shown in Figure (a) below. At time t the charge on the upper sphere is $q(t)$, and the charge on the lower sphere is $-q(t)$. Suppose further that we can drive the charge back and forth through the wire, from one end to other, at a frequency ω: $q(t) = q_o \cos(\omega t)$

(a) Write down the corresponding, oscillating ***electric dipole***: $\boldsymbol{p}(t)$, *in terms of* the charge q_o and the separating distance s.

(b) In the far field approximation $r \gg s$, calculate the ***vector-potential*** of this dipole system and express the vector-potential in terms of $\boldsymbol{p}(t)$.

(c) An electric dipole, $\boldsymbol{p}$, oscillates with a frequency ω and amplitude p_o. It is placed at a distant +a/2 from an infinite perfectly conducting plane and the dipole is parallel to the plane as shown in Figure (b) below. Find the ***electric*** and ***magnetic field*** for distance $r \gg a$.

(Note: for approximation: $r_1 \approx r - a/2 \sin\theta \cos\varphi$; $r_2 \approx r + a/2 \sin\theta \cos\varphi$; $1/r_1 \approx 1/r_2 \approx 1/r$.)



I-10 [10]

Consider an infinitely long, uniform line charge with line charge density λ (measured in Coulomb/meter). The line charge is placed on the *z*-axis of reference frame *S* (*x*, *y*, *z*) and is traveling with relativistic speed $v = 0.9\,c$ in the +*z* direction. Here, *c* is the speed of light.

(a) Calculate the electric field ***E*** and the magnetic field ***B*** of the line charge in frame *S* (*x*, *y*, z).

(b) Consider the reference frame *S*' (*x*', *y*', *z*'). The axes *x'*, *y'* and *z'* of *S*' are parallel to the axis *x*, *y*, z of *S*. The frame *S*' moves relative to the frame *S* with a speed v=0.9c in the +*z* direction. Calculate the electric field ***E'*** and magnetic field ***B***' in frame *S*'(*x*', *y*', *z*').

(c) The line charge density λ (measured in *S*) is not equal to the line charge density λ' (measured in S'). Calculate λ'.

Solution I-1

Throwing a ball from a cliff

Let the inclination angle be θ. Then the horizontal speed is $v_x = v\cos\theta$, and the initial vertical speed is $v_y = v\sin\theta$. The time it takes for the ball to hit the ground is given by $h + (v\sin\theta)t - gt^2/2 = 0$. Therefore,

$$t = \frac{v}{g}\left(\sin\theta + \sqrt{\sin^2\theta + \beta}\right), \quad \text{where} \quad \beta \equiv \frac{2gh}{v^2}. \tag{3.119}$$

(The "−" solution for t from the quadratic formula corresponds to the ball being thrown backward down through the cliff.) The horizontal distance traveled is $d = (v\cos\theta)t$, which gives

$$d = \frac{v^2}{g}\cos\theta\left(\sin\theta + \sqrt{\sin^2\theta + \beta}\right). \tag{3.120}$$

We want to maximize this function of θ. Taking the derivative, multiplying through by $\sqrt{\sin^2\theta + \beta}$, and setting the result equal to zero, gives

$$(\cos^2\theta - \sin^2\theta)\sqrt{\sin^2\theta + \beta} = \sin\theta\big(\beta - (\cos^2\theta - \sin^2\theta)\big). \tag{3.121}$$

Using $\cos^2\theta = 1 - \sin^2\theta$, and then squaring and simplifying this equation, gives an optimal angle of

$$\sin\theta_{\max} = \frac{1}{\sqrt{2+\beta}} \equiv \frac{1}{\sqrt{2 + 2gh/v^2}}. \tag{3.122}$$

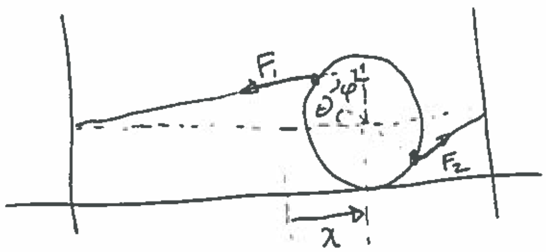
Plugging this into Eq. (3.120), and simplifying, gives a maximum distance of

$$d_{\max} = \frac{v^2}{g}\sqrt{1+\beta} \equiv \frac{v^2}{g}\sqrt{1 + \frac{2gh}{v^2}}. \tag{3.123}$$

If $h = 0$, then $\theta_{\max} = \pi/4$ and $d_{\max} = v^2/g$, in agreement with the example in Section 3.4. If $h \to \infty$ or $v \to 0$, then $\theta_{\max} \approx 0$, which makes sense.

I-2 Solution:

The motion of the disk is confined to the vertical plane. Let the displacement of the center of mass from equilibrium be x and the angular displacement be θ, as shown in the Fig.



To first order in θ, the restoring forces are

$$F_1 = k(l + x - l_0), \quad F_2 = k(l - x - l_0)$$

The equations of motion are then

$$M\ddot{x} = F_2 - F_1 = -2kx$$

$$\ddot{x} + \frac{2k}{M}x = 0 \qquad (1)$$

and $I\ddot{\theta} = (F_2 - F_1)Asin\phi,$

where $I = \frac{1}{2}MA^2$ and ϕ is given by

$$\frac{\sin(\pi - \phi)}{l + A + x} = \frac{\sin(\theta)}{l + x}$$

or

$$\sin(\phi) \approx \frac{l + A}{l} sin\theta \approx \frac{l + A}{l}\theta$$

i.e.

$$\ddot{\theta} + \frac{4k(l - l_0)(l + A)}{MlA}\theta = 0 \qquad (2)$$

The angular frequencies of oscillation are given from eqn (1): $\omega_1 = \sqrt{2k/M}$ (associated with the left right motion of the disk, no rotation)

and eqn (2): $\omega_2 = \sqrt{4k(l - l_0)(l + A)/MlA}$. (associated with the rotation of the disk, fixed position).

$\boxed{I\text{-}3}$ $\mu(x) = kx$ M, L

a) $$M = \int_0^L \mu(x)\,dx = \int_0^L kx\,dx = k\frac{L^2}{2}$$

$$k = \frac{2M}{L^2}$$

b) $$v = \sqrt{\frac{F}{\mu}}$$ $F = \text{const.}$, $\mu(x)$

$$\Rightarrow \quad v(x) = \sqrt{\frac{F}{\mu(x)}} \qquad v(x) = \frac{dx}{dt}$$

$$\frac{dx}{dt} = \sqrt{\frac{F}{\mu(x)}}$$

$$dt = \sqrt{\frac{\mu(x)}{F}}\,dx$$

$$T = \int_0^T dt = \int_0^L \sqrt{\frac{\mu(x)}{F}}\,dx = \sqrt{\frac{2M}{FL^2}} \int_0^L x^{1/2}\,dx = \sqrt{\frac{2M}{FL^2}}\,\frac{2}{3}L^{3/2}$$

$$= \boxed{\sqrt{\frac{8ML}{9F}}}$$

$\boxed{\text{I-4}}$

$$T = T_{trans}^{COM} + T_{rot}$$

$$T_{trans}^{COM} = \frac{1}{2} M \dot{\vec{R}}_{COM} \quad \text{with} \quad M = 2m$$

$$\vec{R}_{COM} = \begin{pmatrix} l \cos\varphi \\ \frac{l}{2} \sin\varphi \end{pmatrix} \qquad \dot{\vec{R}}_{COM} = \begin{pmatrix} -l \sin\varphi \\ \frac{1}{2} l \cos\varphi \end{pmatrix} \dot{\varphi}$$

$$\dot{\vec{R}}^2_{COM} = l^2 \dot{\varphi}^2 \left(1 - \frac{3}{4} \cos^2\varphi\right)$$

$$T_{rot} = 2 T_{rot1} = 2 \, \frac{1}{2} I_{zz} \omega_z^2$$

moment of inertia of rod 1 for rotation around COM of rod 1 is:

$$I_{zz} = \int_V \rho \, (x^2 + y^2) \, dV$$

$$= \rho \int_{-l/2}^{l/2} \int_0^{2\pi} \int_0^R (r^2 \sin^2\theta + z^2) \, r \, dr \, d\theta \, dz$$

$$= \frac{1}{4} M R^2 + \frac{1}{12} M l^2$$

$$T_{rot} = \cancel{2} \cdot \frac{1}{\cancel{2}} \left(\frac{1}{4} M R^2 + \frac{1}{12} M l^2 \right) \dot{\varphi}^2$$

$$T = \frac{1}{4} M R^2 \dot{\varphi}^2 + \frac{1}{12} M l^2 \dot{\varphi}^2 + \frac{1}{2} M l^2 \dot{\varphi}^2 \left(1 - \frac{3}{4} \cos^2\varphi\right)$$

(I-5)

No.

ⓐ Distance: $\Delta d = \underbrace{\lambda}_{\text{wavelength of emitted light}} = \dfrac{c}{f_s}$ #

ⓑ Time between adjacent wavefront:

$$t = \frac{\Delta d}{\underbrace{(c-v)}_{\text{observer moving away from the source}}} = \frac{c/f_s}{(c-v)}$$

$$\therefore t = \frac{1}{(1-\beta) f_s}, \quad \beta \overset{\text{def}}{=} \frac{v}{c}$$ #

ⓒ Due to relativistic time-dilation, the observer will measure this time:

$$\therefore t_0 = \frac{t}{\gamma}, \quad \underbrace{\gamma}_{\text{Lorentz factor}} \overset{\text{def}}{=} \frac{1}{\sqrt{1-\beta^2}}$$ #

ⓓ The corresponding observed frequency f_0:

$$f_0 = \frac{1}{t_0} = \frac{\gamma}{t} = \gamma(1-\beta) f_s$$

$$\therefore f_0 = f_s \sqrt{\frac{1-\beta}{1+\beta}}$$ #

$\boxed{\text{I-6}}$

$$\Delta\phi(x,y)=0 \qquad \begin{aligned}\phi(x,0)&=0\\ \phi(x,a)&=0\\ \phi(b,y)&=\phi_0\\ \phi(-b,y)&=\phi_0\end{aligned}$$

$$\frac{\partial^2\phi(x,y)}{\partial x^2}+\frac{\partial^2\phi(x,y)}{\partial y^2}=0 \qquad (i)$$

using the approach of separation of variables, a solution of (i) is:

$$\phi(x,y)=(Ae^{kx}+Be^{-kx})(C\sin ky+D\cos ky)$$

with k being the separation constant.

from symmetry $\phi(-x,y)=\phi(x,y)$ & $A=B$

rewriting $e^{kx}+e^{-kx}=2\cosh kx$

and $C'=2AC$ as well as $D'=2AD$

$$\phi(x,y)=\cosh kx\,(C'\sin ky+D'\cos ky)$$

$\phi(x,y=0)=0$ and $\phi(x,y=a)=0$ require $D'=0$

and $k=n\pi/a$, so

$$\phi(x,y)=C'\cosh\left(\frac{n\pi x}{a}\right)\sin\frac{n\pi y}{a}.$$

The general solution is

$$\phi(x,y)=\sum_{n=1}^{\infty}C_n\cosh\left(\frac{n\pi x}{a}\right)\sin\left(\frac{n\pi y}{a}\right)$$

I-6 (Cont'd)

with

$$\phi(b,y) = \sum_{n=1}^{\infty} C_n \cosh\left(\frac{n\pi b}{a}\right) \sin\frac{n\pi y}{a} = \phi_0$$

from Fourier analysis

$$C_n \cosh\frac{n\pi b}{a} = \begin{cases} 0 & n \text{ is even} \\ \dfrac{4V_0}{n\pi} & n \text{ is odd} \end{cases}$$

$$\phi(x,y) = \frac{4V_0}{\pi} \sum_{n=1,3,5,\ldots}^{\infty} \frac{1}{n} \frac{\cosh(n\pi x/a)}{\cosh(n\pi b/a)} \sin\frac{n\pi y}{a}$$

(I-7) No.

(a) Maxwell equ.

① $\nabla\cdot\vec{E} = 0$

② $\nabla\cdot\vec{B} = 0$

③ $\nabla\times\vec{E} = -\dfrac{\partial\vec{B}}{\partial t}$

④ $\nabla\times\vec{B} = (\mu_0\varepsilon_0)\dfrac{\partial\vec{E}}{\partial t}$ #

(b) Attempt to combine equ-(3) and -(4):

$$\nabla\times(\nabla\times\vec{E}) = -\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{B}) = -(\mu_0\varepsilon_0)\frac{\partial^2\vec{E}}{\partial t^2}$$

$$\nabla\times(\nabla\times\vec{B}) = (\mu_0\varepsilon_0)\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{E}) = -(\mu_0\varepsilon_0)\frac{\partial^2\vec{B}}{\partial t^2}$$

$$\vec{\nabla}\times(\nabla\times\vec{E}) = \nabla(\vec{\nabla}\cdot\vec{E}) - \nabla^2 E = -\nabla^2\vec{E} \quad (\because \vec{\nabla}\cdot\vec{E} = 0)$$

$$\vec{\nabla}\times(\nabla\times\vec{B}) = \nabla(\vec{\nabla}\cdot\vec{B}) - \vec{\nabla}^2 B = -\nabla^2\vec{B} \quad (\because \vec{\nabla}\cdot\vec{B} = 0)$$

⑤ $\therefore\ \nabla^2\vec{E} = (\mu_0\varepsilon_0)\dfrac{\partial^2 E}{\partial^2 t}$ #

⑥ $\therefore\ \vec{\nabla}^2\vec{B} = (\mu_0\varepsilon_0)\dfrac{\partial^2 B}{\partial t^2}$ #

Wave equations for $\vec{E}$, $\vec{B}$.

(1-7)

(c) Given: $\vec{E} = \vec{E}_0 \cos(\vec{k}\cdot\vec{r} - \omega t)$, $\vec{B} = \vec{B}_0 \cos(\vec{k}\cdot\vec{r} - \omega t)$ } plane-wave solution

$\frac{\omega}{k} = c$ speed-of-light

From equ. (1) and (2):

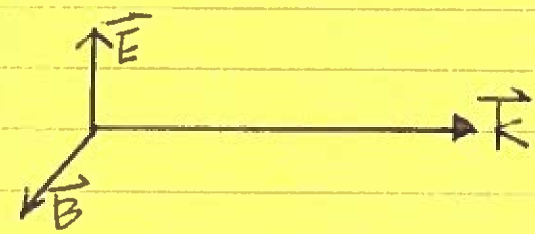
$$\vec{\nabla}\cdot\vec{E} = \partial_x E_x + \partial_y E_y + \partial_z E_z = k_x E_x + k_y E_y + k_z E_z$$

$$\vec{\nabla}\cdot\vec{B} = \partial_x B_x + \partial_y B_y + \partial_z B_z = k_x B_x + k_y B_y + k_z B_z$$

(7) $\therefore \vec{k}\cdot\vec{E} = \vec{\nabla}\cdot\vec{E} = 0$ #

(8) $\therefore \vec{k}\cdot\vec{B} = \vec{\nabla}\cdot\vec{B} = 0$ #

$\vec{k}, \vec{E}, \vec{B}$ are perpendicular to each other.



(d) from equ (3) and (4):

(9) $\vec{k}\times\vec{E} = -\omega\vec{B}$, $kE = \omega B$, $\frac{E}{B} = \left(\frac{\omega}{k}\right)$

(10) $\vec{k}\times\vec{B} = \mu_0\varepsilon_0\omega\vec{E}$, $kB = \mu_0\varepsilon_0\omega E$, $\frac{E}{B} = \left(\frac{k}{\omega}\right)\left(\frac{1}{\mu_0\varepsilon_0}\right)$

From Equ. (9) and (10): $\left(\frac{\omega}{k}\right)^2 = \frac{1}{\varepsilon_0\mu_0}$

$$\therefore \underline{c^2 = \frac{1}{\varepsilon_0\mu_0}} \ \#$$

1-8 Solution:

(a) Using cylindrical coordinates(r, θ, z), we have $P = P_r = \frac{P_0 r}{2}$. The bound charge density is

$$\rho = \nabla \cdot \boldsymbol{P} = -\frac{1}{r}\frac{\partial}{\partial r}\left(r \cdot \frac{P_0 r}{2}\right) = -P_0$$

(b) As $\boldsymbol{\omega} = \omega \hat{z}$, the volume current density at a point $\boldsymbol{r} = r\hat{r} + z\hat{z}$ in the cylinder is,

$\boldsymbol{j}(r) = \rho \boldsymbol{v} = \rho \boldsymbol{\omega} \times \boldsymbol{r} = -P_0 \omega \hat{z} \times (r\hat{r} + z\hat{z}) = -P_0 \omega r \hat{\theta}$.

On the surface of the cylinder there is also a surface charge distribution of density,

$\sigma = \boldsymbol{n} \cdot \boldsymbol{P} = \hat{r} \cdot \frac{P_0 r}{2}|_{r=a} = \frac{P_0 a}{2}$, producing a surface current density of $\boldsymbol{\alpha} = \sigma \boldsymbol{v} = \frac{P_0}{2} \omega a^2 \hat{\theta}$.

To find the magnetic field at a point on the axis of the cylinder not too near its ends, as the cylinder is very long we can take this point as the origin and regard the cylinder as infinitely long. Then the magnetic induction at the origin is given by

$$\boldsymbol{B} = -\frac{\mu_0}{4\pi}\left(\int_V \frac{j(r') \times r'}{r'^3} dV' + \int_S \frac{\alpha(r') \times r'}{r'^3} dS'\right)$$

$$\int_V \frac{j(r') \times r'}{r'^3} dV' = P_0 \omega \int_0^{2\pi} d\theta \int_0^a r dr \int_{-\pi/2}^{\pi/2} cos\beta d\beta\, \hat{z} = 2\pi P_0 \omega a^2 \hat{z}$$

, where $\beta = \text{atan}(z/r)$.

$$\int_S \frac{\alpha(r') \times r'}{r'^3} dS' = -\frac{P_0}{2}\omega \int_S \frac{d\theta dz}{(a^2 + z^2)^{\frac{3}{2}}} \hat{z} = -2\pi P_0 \omega a^2 \hat{z}\,.$$

Hence, the magnetic induction B vanishes at points of the cylindrical axis not too near the ends.

(I-9)

(a) Dipole $= \vec{P} \overset{\text{def}}{=} q\vec{S} = \underbrace{(q_0 S)}_{P_0} \cos\omega t\ \hat{k}$

Current $= \vec{I} \overset{\text{def}}{=} \frac{dq}{dt} = -(q_0 \omega) \sin\omega t\ \hat{k}$

(b) For radiation, $t \to t' = t - (r/c)$

Each point along the wire contributes

to radiation field at r.

$\therefore$ Vector pot. $\vec{A} = \frac{\mu_0}{4\pi} \int_{-S/2}^{S/2} \frac{-(q_0\omega)\sin\omega(t - r/c)}{r} dz\ \hat{k}$

$$\vec{A} \simeq \frac{\mu_0}{4\pi} \frac{-(q_0 S\omega)\sin\omega(t - r/c)}{r} \hat{k}$$

$$\therefore\ \vec{A} \simeq \frac{\mu_0}{4\pi}\left(\frac{\dot{\vec{P}}(t)}{r}\right) \# \quad , k \overset{\text{def}}{=} \omega/c .$$

(c) $\vec{A}_{total} = \vec{A}_1 + \vec{A}_2$

$$= \frac{\mu_0}{4\pi}\left(\frac{\dot{\vec{P}}_1(t)}{r_1} + \frac{\dot{\vec{P}}_2(t)}{r_2}\right)$$

$$\vec{A}_{total} \cong -i \frac{\mu_0}{4\pi}(\omega P_0) e^{-i\omega t}\left(\frac{e^{ikr_1}}{r_1} - \frac{e^{ikr_2}}{r_2}\right) \hat{z}$$

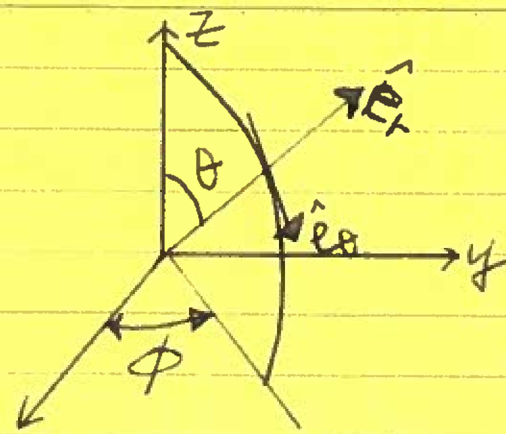
$$\cong -i \frac{\mu_0}{4\pi}\left(\frac{\omega P_0}{r}\right) e^{-i\omega t}\left[e^{ik\frac{a}{2}\sin\theta\cos\phi} - e^{-ik\frac{a}{2}\sin\theta\cos\phi}\right]$$

$$\therefore\ \vec{A}_{total} \cong \underbrace{\frac{\mu_0}{2\pi}\left(\frac{\omega P_0}{r}\right)}_{\text{Strength}} \underbrace{e^{-i(\omega t - kr)}}_{\substack{t\text{-dependent} \\ r\text{-dependent}}} \underbrace{\sin\left[\frac{ka}{2}\sin\theta\cos\phi\right]}_{\text{Angular-dependent}} \#$$

(I-9)

No.

(c)



$$\hat{e}_z = \cos\theta\, \hat{e}_r - \sin\theta\, \hat{e}_\theta$$

Magnetic field: $\vec{B} = \vec{\nabla} \times \vec{A}$

$$\vec{B} \cong \frac{1}{r}\frac{\partial}{\partial r}(r A_\theta)\, \hat{e}_\phi$$

$$\therefore \vec{B} \cong \hat{e}_\phi \frac{i\, \omega_0^2\, p_0}{2\pi \varepsilon_0 c^3} \left(\frac{e^{i(kr-\omega t)}}{r}\right) \sin\theta\, \sin\left[\frac{ka}{2}\sin\theta\cos\phi\right]$$

#

Electric field: $\vec{E} = c\, \vec{B} \times \hat{e}_r$

$$\therefore \vec{E} \cong \hat{e}_\theta \frac{i\, \omega_0^2\, p_0}{2\pi \varepsilon_0 c^2} \underbrace{\left(\frac{e^{i(kr-\omega t)}}{r}\right)}_{\text{spherical wave}} \underbrace{\sin\theta\, \sin\left[\frac{ka}{2}\sin\theta\cos\phi\right]}_{\text{angular dependence}}$$

#

Solution I-10

The moving line charge constitutes a current $I = \lambda v$ along the z axis, so our problem is to find the combined fields of a line charge and a line current. Let us recognize first that this can be done by elementary methods, without leaving the frame $\mathcal{S}$: Using Gauss's law we can show that the E field of the line charge is $E = 2k\lambda/\rho$ radially outward from the z axis. (Here $k = 1/4\pi\epsilon_o$ is the Coulomb force constant and ρ is the perpendicular distance from the z axis, that is, the first of the coordinates ρ, ϕ, z of cylindrical polar coordinates.) Similarly, using Ampere's law, we can show the B field of the current is $B = (\mu_o/2\pi)I/\rho$ in the direction given by the right-hand rule, where μ_o is the so-called permeability of space. We can express these two well-known results compactly using the unit vectors of cylindrical polar coordinates:

$$\mathbf{E} = \frac{2k\lambda}{\rho}\hat{\rho} \qquad \text{and} \qquad \mathbf{B} = \frac{\mu_o}{2\pi}\frac{I}{\rho}\hat{\boldsymbol{\phi}}. \tag{15.147}$$

Both of these fields are sketched in Figure 15.16(a).

While the derivation using Gauss's and Ampere's laws is perfectly straightforward, it is instructive to rederive the same results by transforming to a frame $\mathcal{S}'$ traveling with the charges. In $\mathcal{S}'$, there is no current, so the only field is the radial electric field, $E' = 2k\lambda'/\rho'$, as shown in Figure 15.16(b). This field is in the direction of the unit vector $\hat{\rho}' = (x'/\rho', y'/\rho', 0)$, so can be written as

$$\mathbf{E}' = \frac{2k\lambda'}{\rho'}\hat{\rho}' = \frac{2k\lambda'}{\rho'^2}(x', y', 0). \tag{15.148}$$

Before we transform this back to the original frame $\mathcal{S}$, we must recognize that the charge densities λ and λ' are not equal: The total charge contained in any given segment of the z axis must be the same in either frame (invariance of charge), so that $\lambda\,\Delta z = \lambda'\Delta z'$, but, because of length contraction, $\Delta z = \Delta z'/\gamma$. Therefore

$$\lambda = \gamma\lambda'. \tag{15.149}$$

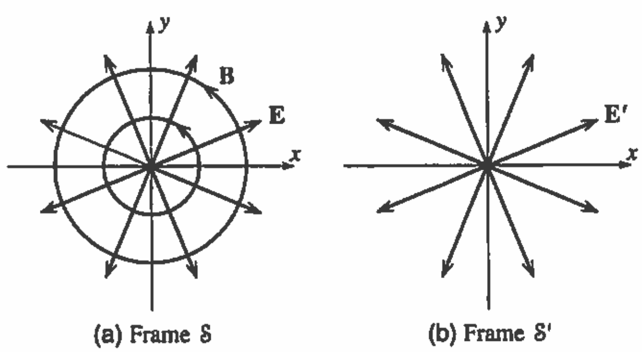


Figure 15.16 The fields produced by a line charge on the z axis. (a) In frame $\mathcal{S}$, the line charge is moving up, out of the page. This constitutes a current, which produces a B field looping around the z axis — in addition to the E field, which is radially out from the z axis. (b) The frame $\mathcal{S}'$ is the rest frame of the charges, so there is no current and hence no B field — just the radial E field.

Physics PhD Qualifying Examination
Part II – Friday, January 23, 2015

Name: **(problems & solutions)**
(please print)
Identification Number:________

STUDENT: **insert a check mark in the left boxes to designate the problem numbers that you are handing in for grading.**
PROCTOR: **check off the right hand boxes corresponding to the problems received from each student. Initial in the right hand box.**

	1	
	2	
	3	
	4	
	5	
	6	
	7	
	8	
	9	
	10	

Student's initials # problems handed in: Proctor's initials

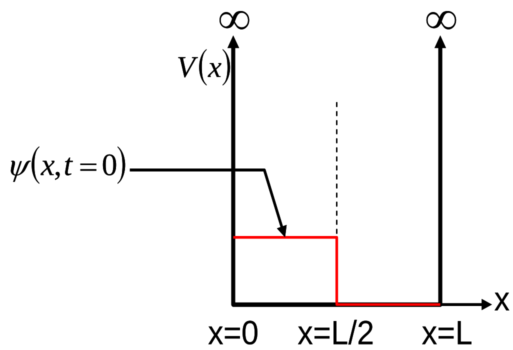
INSTRUCTIONS FOR SUBMITTING ANSWER SHEETS

1. DO NOT PUT YOUR NAME ON ANY ANSWER SHEET. EXAMS WILL BE COLLATED AND GRADED BY THE ID NUMBER ABOVE.
2. Use at least one separate preprinted answer sheet for each problem. Write on only one side of each answer sheet.
3. Write your **identification number** listed above, in the appropriate box on the preprinted sheets.
4. Write the **problem number** in the appropriate box of each preprinted answer sheet. If you use more than one page for an answer, then number the answer sheets with both problem number and page (e.g. Problem 9 – Page 1 of 3).
5. Staple together all the pages pertaining to a given problem. Use a paper clip to group together all problems that you are handing in.
6. A passing distribution for the individual components will normally include at least four passed problems (from problems 1-6) for Quantum Physics and two problems (from problems 7-10) for Thermodynamics and Statistical Mechanics.
7. **YOU MUST SHOW ALL YOUR WORK.**

II-1 [10]

Consider a particle in a one-dimensional potential well of width L with infinitely high walls, such that:

$V(x)=0$, if $0<x<L$;
$V(x)=\infty$, otherwise



At t=0, the wavefunction is:
$\psi(x,0)=\sqrt{2/L}$ for $0<x<L/2$;
$\psi(x,0)=0$, otherwise.

(a) What are the eigenfunctions $\phi_n(x)$ of the particle?

(b) At time $t>0$, what is the probability of finding the particle in state $\phi_n(x)$?

(c) Argue, from symmetry, that certain set of eigenstates has zero probability at all time.

II-2 [10]

A charged particle is bound in a harmonic oscillator potential $V = \frac{1}{2}kx^2$. The system is placed in an external electric field E that is constant in space and time. Calculate the shift of the energy of the ground state to order E^2.

II-3 [10]

A spin ½ particle is in the state

$$|\alpha\rangle = \frac{1}{\sqrt{6}}\begin{pmatrix} 1+i \\ 2 \end{pmatrix} \ ,$$

written in the S_z -representation.

(a) What is the probability of getting $+\hbar/2$ if we measure S_x?

(b) What is expectation value of measuring S_x-in this state?

II-4 [10]

The Yukawa potential has the form $V(r) = \beta \frac{\exp(-\mu r)}{r}$ with constants μ >0 and β >0. The potential describes the binding forces in an atomic nucleus.

Using the Born approximation, calculate the amplitude $f\theta)$ of a wave scattered from $V(r)$ and the total scattering cross section σ. Consider scattering in the forward direction only.

II-5 [10]

Assuming scattering energies are low enough to be dominated by the s partial wave, determine the scattering cross section for a particle by a potential given by:

$$V = -V_0 \quad \text{for } (r < a)$$
$$V = 0 \quad \text{for } (r > a).$$

II-6 [10]

A hydrogen atom in its ground state is placed between the parallel plates of a capacitor. At time $t < 0$, the electric potential difference between the capacitor plates is zero. Starting at time $t = 0$, the electric potential of the capacitor plates changes. Hence, a spatially uniform but time-dependent electric field is established between the capacitor plates. The electric field is described by $E(t) = E_0 \exp(-t/\tau)$ with $\tau = constant$.

(a) Calculate the probability that the electron is excited to the 2s energy state as a result of the perturbation.

(b) Calculate the probability that the electron is excited to the 2p energy state as a result of the perturbation.

Hint:

Normalized hydrogen eigenfunctions	**Energy, Ry**
$\psi_{1,0,0} = \pi^{-1/2} a_0^{-3/2} e^{-r/a_0}$	-1
$\psi_{2,0,0} = (8\pi)^{-1/2} a_0^{-3/2} \left(1 - \dfrac{r}{2a_0}\right) e^{-r/2a_0}$	$-\frac{1}{4}$
$\psi_{2,1,-1} = (64\pi)^{-1/2} a_0^{-5/2} (x - iy) e^{-r/2a_0}$	$-\frac{1}{4}$
$\psi_{2,1,0} = (32\pi)^{-1/2} a_0^{-5/2} z e^{-r/2a_0}$	$-\frac{1}{4}$
$\psi_{2,1,1} = (64\pi)^{-1/2} a_0^{-5/2} (x + iy) e^{-r/2a_0}$	$-\frac{1}{4}$

II-7 [10]

The equation of state of a simple ferromagnetic material is given by the implicit expression

$$m = \tanh\left(\frac{Jm + B}{kT}\right),$$

where $m = m(T, B)$ is the dimensionless magnetization (order parameter), B is the external magnetic field, T is the temperature, k is the Boltzmann constant, and J is a material-specific constant.

(a) What is the *critical temperature* T_c below which the system exhibits spontaneous magnetization? (We refer to spontaneous magnetization when $m \neq 0$ at $B = 0$.)

(b) Show that in the region just below T_c, the spontaneous magnetization behaves as

$$m(T,0) \approx \text{const.}\,|\,T - T_c\,|^b \ ,$$

and determine the value of the *critical exponent* b.

II-8 [10]

Initially, an ideal gas is confined to a volume V_i. The initial temperature of the ideal gas is T_i. Subsequently, the ideal gas is adiabatically expanded to a volume V_f. Calculate the final temperature T_f of the ideal gas.

II-9 [10]

Consider an ensemble of N distinguishable classical one–dimensional harmonic oscillators with identical frequency ω and mass m at temperature T.

(a) What is the entropy $S(T, N)$ of the system?

Hint: Start with the proper classical single-particle partition function:

$$Z = \int \frac{dpdq}{h} e^{-\beta H(p,q)},$$

where $H(p,q)$ is the classical single-particle Hamiltonian and h is Planck's constant.

II-10 [10]

The rotational energy levels of a diatomic molecule are described by:

$\varepsilon_l = \frac{\hbar}{2I} l(l+1)$ with quantum number $l=0, 1, 2, \ldots$ and moment of inertia I. The degeneracy factor of the energy levels is $g_l=2l+1$.

- **(a)** Calculate the partition function for the diatomic molecule.
- **(b)** Calculate the average energy of the diatomic molecule.
- **(c)** Calculate internal energy and the specific heat capacity at constant volume of an ensemble of N diatomic molecules for very high temperatures T.

(II-1) No.

ⓐ $H|\phi_n\rangle = E_n|\phi_n\rangle$

$\therefore \phi_n(x) = \sqrt{\frac{2}{L}}\, \sin\left(n\pi \frac{x}{L}\right)$, $n = 1, 2, 3, \cdots$

$E_n = \frac{\hbar^2}{2m}\left(\frac{n\pi}{L}\right)^2$

ⓑ Initial wavefct: $\psi(x,0) = \begin{cases} \sqrt{\frac{2}{L}}, & 0 < x < L/2 \\ 0, & 0 \end{cases}$

$|\psi\rangle = a_n|\phi_n\rangle$

$\therefore a_n = \langle \phi_n|\psi(0)\rangle = \int_0^{L/2} \sqrt{\frac{2}{L}}\sqrt{\frac{2}{L}}\, \sin\left(n\pi\frac{x}{L}\right) dx$

$a_n = \frac{-2}{n\pi}\left[\cos n\pi\frac{x}{L}\right]_0^{L/2} = \frac{2}{n\pi}\left[\cos\frac{n\pi}{2} - 1\right]$

$a_n = \begin{cases} \frac{+2}{n\pi}, & n = 1, 3, 5, 7, \cdots \text{ (odd)} \\ \frac{4}{n\pi}, & n = 2(p+1), \ p = 0, 1, 2, 3, \cdots \\ 0, & n = 4p, \ p = 0, 1, 2, 3, \cdots \end{cases}$

#

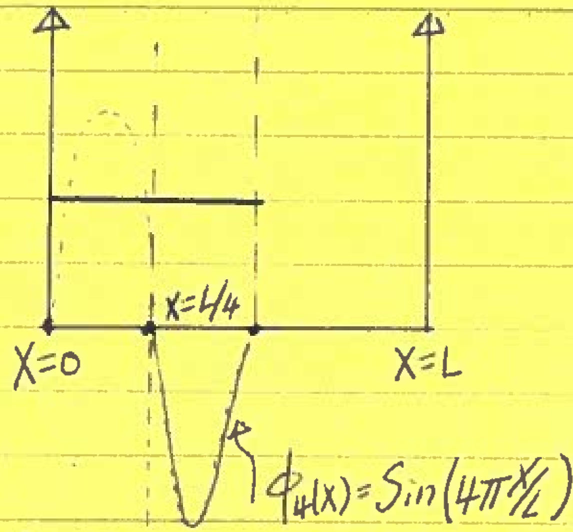
ⓑ Wavefct at time t: $\psi(x,t) = \sum_n a_n e^{-i/\hbar E_n t}|\phi_n\rangle$

Probability: $P_n(t) = |a_n e^{-i/\hbar E_n t}|^2 = a_n^2$ #

No.

(II-1)

(c)



for $n=4$, ϕ_n is an odd function w.r.t. $x=L/4$,

$\therefore$ integration has an exact cancellation.

$\therefore$ $a_4=0$ #

The same argument applies to $n=4,8,12,\ldots$

$\therefore$ $a_n=0$, $n=4p$, $p=1,2,3,\ldots$ #

II-2 Solution:

Take the direction of the electric field as the x-direction. The Hamiltonian of the system is

$$H = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}kx^2 - qEx = H_0 + H',$$

Where $H' = -qEx$ is to be treated as a perturbation.

The wave function of the ground state of a harmonic oscillator is $\psi(x) = \sqrt{\frac{\alpha}{\pi^{1/2}}}e^{-\frac{1}{2}\alpha^2x^2}$, where

$$\alpha = \sqrt{\frac{m\omega}{\hbar}}, \omega = \sqrt{\frac{k}{m}}.$$

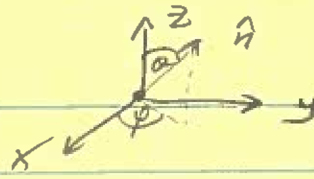
As ψ_0 is an even function, the first order correction vanishes, and we have to go to the second order,

$\langle n'|x|n\rangle = \frac{1}{\alpha}(\sqrt{n'/2}\delta_{n,n'-1} + \sqrt{(n'+1)/2}\delta_{n,n'+1})$, and hence

$H'_{0,n} = -qE\langle 0|x|n\rangle = -qE/(\sqrt{2}\alpha)\delta_{n,1}$ and

$$\Delta E_0^2 = \sum_n' \frac{|H'_{0,n}|^2}{E_0^0 - E_n^0} = \sum_n' \frac{\frac{q^2E^2}{2\alpha^2}\delta_{n,1}}{-n\hbar\omega} = -\frac{q^2E^2}{2\hbar\omega\alpha^2} = -\frac{q^2E^2}{2m\omega^2}$$

$\boxed{\text{II}-3}$



Recall:

$$|S_{\hat{n}};+\rangle = |\vec{S}\cdot\hat{n};+\rangle = \cos(\vartheta/2)|+\rangle + \sin(\vartheta/2)e^{i\varphi}|-\rangle$$

$$|S_{\hat{n}};-\rangle = |\vec{S}\cdot\hat{n};-\rangle = \sin(\vartheta/2)|+\rangle - \cos(\vartheta/2)e^{i\varphi}|-\rangle$$

where $|+\rangle \equiv |S_z;+\rangle$, $|-\rangle \equiv |S_z;-\rangle$

S_x, $\varphi = 0, \vartheta = \pi/2$

$$|S_x;+\rangle = \frac{1}{\sqrt{2}}|+\rangle + \frac{1}{\sqrt{2}}|-\rangle \doteq \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$$

$$|\alpha\rangle \doteq \chi = \frac{1}{\sqrt{6}}\begin{pmatrix}1+i\\2\end{pmatrix}$$

(a) $$\langle S_x;+|\alpha\rangle = \frac{1}{\sqrt{12}}\{1\cdot(1+i) + 1\cdot 2\} = \frac{1}{\sqrt{12}}(3+i)$$

$$P(S_x = +\hbar/2;\alpha) = |\langle S_x;+|\alpha\rangle|^2 = \frac{1}{12}(9+1) = \frac{10}{12} = \boxed{\frac{5}{6}}$$

clearly, $P(S_x = +\hbar/2;\alpha) + P(S_x = -\hbar/2;\alpha) = 1$ ■

thus, $P(S_x = -\hbar/2;\alpha) = 1 - P(S_x = +\hbar/2;\alpha) = 1 - 5/6 = 1/6$

(b) $$\langle S_x\rangle_\alpha = P(S_x = +\hbar/2;\alpha)\frac{\hbar}{2} + P(S_x = -\hbar/2;\alpha)\left(-\frac{\hbar}{2}\right)$$

$$= \frac{5}{6}\cdot\frac{\hbar}{2} + \frac{1}{6}\left(-\frac{\hbar}{2}\right) = \frac{\hbar}{2}\cdot\frac{4}{6} = \boxed{\frac{\hbar}{3}}$$ ■

II-3 (contd)

Alternatively: recall: $S_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$$\langle S_x \rangle_\alpha = \langle \alpha | S_x | \alpha \rangle = \frac{1}{\sqrt{6}}(1-i, 2)\frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\frac{1}{\sqrt{6}}\begin{pmatrix} 1+i \\ 2 \end{pmatrix}$$

$$= \frac{\hbar}{12}(1-i, 2)\begin{pmatrix} 2 \\ 1+i \end{pmatrix} = \frac{\hbar}{12}\{(1-i)\cdot 2 + 2\cdot(1+i)\}$$

$$= \frac{\hbar}{12}4 = \boxed{\frac{\hbar}{3}}$$

∎

II-4 [10]

The Yukawa potential has the form $V(r) = \beta \frac{\exp(-\mu r)}{r}$ with constants $\mu > 0$ and $\beta > 0$.

The potential describes the binding forces in an atomic nucleus.
Using the Born approximation, calculate the amplitude $f(\theta)$ of a wave scattered from $V(r)$ and the total scattering cross section σ. Consider scattering in the forward direction only.

II-4

$$F(\theta) = \frac{-2m}{\hbar^2 k} \int_0^\infty r V(r) \sin(Kr)\, dr \quad \text{with } K = 2k \sin\frac{\theta}{2}$$

$$= -\frac{2m}{\hbar^2 k} \int_0^\infty r \beta \frac{e^{-\mu r}}{r} \sin(Kr)\, dr$$

$$= -\frac{2m\beta}{\hbar^2(\mu^2 + K^2)}$$

$$\sigma = \iint |F(\theta)|^2 \sin\theta \, d\theta \, d\varphi$$

$$\sigma = 2\pi \left(\frac{2m\beta}{\hbar^2}\right)^2 \frac{1}{\mu^4} \left(\frac{\mu}{k}\right)^2 \int_0^{\frac{2k}{\mu}} \frac{x}{[1+x^2]} \, dx$$

$$\text{with } x = \frac{2k}{\mu} \sin\frac{\theta}{2}$$

$$\sigma = \pi \left(\frac{4m\beta}{\mu \hbar^2}\right)^2 \frac{1}{(\mu\hbar)^2 + 8mE} \quad \text{with } E = \frac{\hbar^2 k^2}{2\mu}$$

II-5 Solution:

The radial Schrodinger equation can be written in the form

$$\chi_l'' + \left(k^2 - \frac{l(l+1)}{r^2}\right)\chi_l(r) = 0 \quad (r > a),$$

$$\chi_l'' + \left(k'^2 - \frac{l(l+1)}{r^2}\right)\chi_l(r) = 0 \quad (r < a),$$

Where $\chi = rR(r)$,

$$k = \frac{2mE}{\hbar^2}, \qquad k' = \frac{2m(E+V_0)}{\hbar^2}.$$

Considering the s partial wave ($l = 0$), these become,

$$\chi_l'' + k^2\chi_l(r) = 0 \quad (r > a),$$

$$\chi_l'' + k'^2(r) = 0 \quad (r < a),$$

With solutions

$$\chi_l = A\sin(k'r), \ \ (r < a)$$

$$\chi_l = \sin(kr + \delta_0), \ \ (r > a).$$

The continuity condition $(ln\chi_l)'|_{r=a^-} = (ln\chi_l)'|_{r=a^+}$ gives,

$$k\tan(k'a) = k'\tan(ka + \delta_0),$$

$$\rightarrow \delta_0 = \operatorname{atan}\left(\frac{k}{k'}\tan(k'a)\right) - ka$$

For low energies, $k \rightarrow 0, k' \rightarrow k_0 = \sqrt{\frac{2mV_0}{\hbar^2}}$

and $\delta_0$becomes,

$$\delta_0 \approx ka(\frac{\tan(k_0a)}{k_0a} - 1)$$

The total scattering cross section is then

$$\sigma \approx \frac{4\pi}{k^2}\sin^2\delta_0 \approx \frac{4\pi}{k^2}\delta_0^2 = 4\pi a^2\left(\frac{\tan(k_0a)}{k_0a} - 1\right)^2$$

If $k_0a \ll 1$,

$$\sigma \approx 4\pi a^2\left(\frac{k_0a}{k_0a} + \frac{(k_0a)^3}{3k_0a} - 1\right)^2 = 16\pi a^6m^2V_0^2/9\hbar^4$$

Solution II- 6

For time-dependent perturbations a general wave function is

$$\Psi(\vec{r},t) = \sum_j a_j(t)\,\psi_j(\vec{r})\,e^{-i\omega_j t} \tag{1}$$

Where the ψ_j satisfy

$$H_0\,\psi_j = \hbar\omega_j\,\psi_j \tag{2}$$

For the time-dependent perturbation $V(t)$,

$$V(t) = -e\,|\vec{E}_0|\,z\,e^{-t/\tau} \tag{3}$$

From Schroedinger's equation we can derive an equation for the time development of the amplitudes $a_j(t)$:

$$i\hbar\frac{\partial}{\partial t}\Psi = [H_0 + V(t)]\,\Psi$$

$$i\hbar\frac{\partial}{\partial t}a_j(t) = \sum_\ell a_\ell \langle j|V(t)|\ell\rangle\, e^{it(\omega_j-\omega_\ell)} \tag{5}$$

If the system is initially in the ground state, we have $a_{1S}(0) = 1$ and then other values of $a_j(0)$ are zero. For small perturbations it is sufficient to solve the equation for $j \neq 1S$:

$$\frac{\partial}{\partial t}a_j(t) = \frac{ie\,|\vec{E}_0|}{\hbar}\langle j|z|1S\rangle\, e^{-t\left[\frac{1}{\tau} - i(\omega_j-\omega_{1S})\right]}$$

$$a_j(\infty) = \frac{ie\,|\vec{E}_0|\langle z\rangle}{\hbar}\int_0^\infty dt\, e^{-t\left[\frac{1}{\tau} - i(\omega_j-\omega_{1S})\right]}$$

$$= \frac{ie\,|\vec{E}_0|\,\langle z\rangle\,\tau}{\hbar\left[1 - i\tau(\omega_j-\omega_{1S})\right]}$$

The general probability P_j that a transition is made to state j is given by

$$P_j = |a_j(\infty)|^2 = \frac{(e\,|\vec{E}_0|\,\tau)^2\langle j|z|1S\rangle^2}{\hbar^2\left[1+\tau^2(\omega_j-\omega_{1S})^2\right]}$$

II-6 (cont'd)

This probability is dimensionless. It should be less than unity for t this theory to be valid.

a) For the state $j = 2S$ the probability is zero. It vanishes because the matrix element of z is zero: $\langle 2S|z|1S\rangle = 0$ because of parity. Both S-states have even parity, and z has odd parity.

b) For the state $j = 2P$ the transition is allowed to the $L = 1$, $M = 0$ orbital state, which is called $2P_z$. The matrix element is similar to the problem of the Stark effect. The $2P$ eigenstate for $L = 1$, $S = 0$ is in Eqn. 5 and that for the $1S$ state is

$$\frac{e^{-r/a_0}}{\sqrt{\pi a_0^3}}$$

The integral is

$$\langle 2P_z|z|1S\rangle = \frac{2\pi}{\pi a_0^4\sqrt{32}}\int_0^\infty dr\, r^4 e^{-3r/2a_0}\int_0^\pi d\theta \sin\theta\cos^2\theta$$

$$= \frac{1}{3\sqrt{2}\, a_0^4}\int_0^\infty dr\, r^4 e^{-3r/2a_0} = a_0\left(\frac{2^{3/2}}{3}\right)^5$$

Where a_0 is the Bohr radius of the hydrogen atom.

$\boxed{\text{II-7}}$

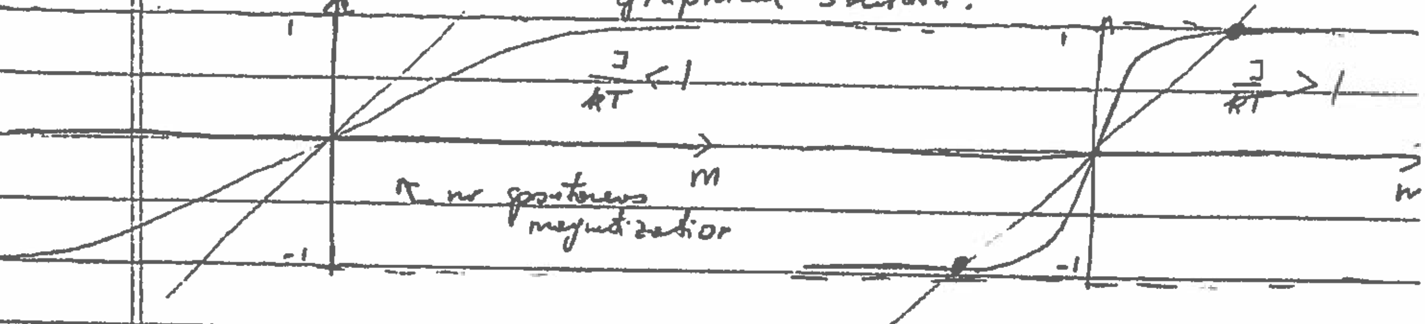
$\boxed{\text{II-[redacted]7}}$

$$m = \tanh\left(\frac{Jm + B}{kT}\right)$$

spontaneous magnetization: $m \neq 0$ when $B = 0$

a) $m = \tanh\left(\frac{J}{kT} m\right)$ $\qquad \tanh(x) = x - \frac{1}{3}x^3 \pm$

"graphical" solution:



For $\frac{J}{kT} > 1$ $\left(T < \frac{J}{k}\right)$ there are 2 non-trivial solutions corresponding to spontaneous magnetization.
Thus, the critical temperature: $\frac{J}{kT_c} = 1$

$$\boxed{T_c = \frac{J}{k}}$$

b) for $T \lesssim T_c$: $\qquad (B = 0)$

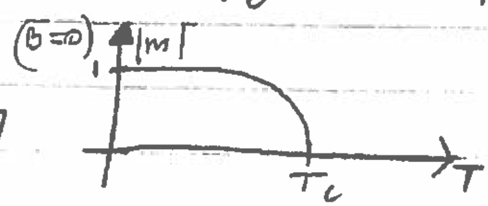
$$m \simeq \frac{J}{kT} m - \frac{1}{3}\left(\frac{J}{kT}\right)^3 m^3 \qquad (m \neq 0)$$

$$1 = \frac{T_c}{T} - \frac{1}{3}\left(\frac{T_c}{T}\right)^3 m^2$$

$$m^2 = 3\frac{T^3}{T_c^3}\left(\frac{T_c}{T} - 1\right) = 3\frac{T^3}{T_c^3}\,\frac{T_c - T}{T} \simeq 3\,\frac{T_c - T}{T_c} \quad \text{for } \frac{T_c - T}{T_c} \ll 1$$

$$m = \pm\sqrt{3}\sqrt{\frac{T_c - T}{T_c}} \sim \pm (T_c - T)^{1/2}$$

$$\boxed{\beta = 1/2}$$



II-8 Solution

$$\delta Q = 0$$

$$0 = \nu c_V \, dT + p\,dV$$

ideal gas $\quad pV = \nu R T$

or $\quad p\,dV + V\,dp = \nu R\,dT$

$$\frac{p\,dV}{\nu R} + \frac{V\,dp}{\nu R} = dT$$

$$0 = \nu c_V \left[\frac{p\,dV}{\nu R} + \frac{V\,dp}{\nu R}\right] + p\,dV$$

$$0 = \frac{c_V}{R}\left(p\,dV + V\,dp\right) + p\,dV$$

$$0 = \left(\frac{c_V}{R} + 1\right) p\,dV + \frac{c_V}{R} V\,dp$$

$$0 = \left(\frac{c_V}{R} + 1\right)\frac{dV}{V} + \frac{c_V}{R}\frac{dp}{p}$$

$$0 = \frac{R}{c_V}\left(\frac{c_V}{R} + 1\right)\frac{dV}{V} + \frac{dp}{p}$$

$$0 = \left(1 + \frac{R}{c_V}\right)\frac{dV}{V} + \frac{dp}{p}$$

$$0 = \gamma \frac{dV}{V} + \frac{dp}{p} \qquad \gamma = \frac{c_p}{c_V} = \frac{c_V + R}{c_V}$$

$$\text{const} = \gamma \ln V + \ln p$$

$$pV^{\gamma} = \text{const} \qquad \text{or} \quad p = \frac{\nu R T}{V}$$

$$T V^{\gamma - 1} = \text{const.} \qquad T_1 V_1^{\gamma-1} = T_2 V_2^{\gamma-1} \;\Rightarrow\; T_f = T_i \frac{V_1^{\gamma-1}}{V_2^{\gamma-1}}$$

$\boxed{\text{II-9}}$

Classical ensemble of oscillators of identical frequency

$$H = \frac{p^2}{2m} + \frac{m\omega^2}{2} q^2$$

$$Z = \int \frac{dp\,dq}{h} e^{-\beta H(p,q)} = \frac{1}{h} \int dp\, e^{-\frac{\beta}{2m} p^2} \int dq\, e^{-\frac{\beta m \omega^2}{2} q^2}$$

$$= \frac{1}{h} \sqrt{\frac{\pi}{\beta/2m}} \cdot \sqrt{\frac{\pi}{\frac{\beta m \omega^2}{2}}} = \frac{1}{h} \frac{\pi}{\sqrt{\beta^2 \omega^2/4}} = \frac{2\pi}{h \omega \beta}$$

$$= \frac{1}{\hbar \omega \beta} = \boxed{\frac{kT}{\hbar \omega}}$$

$$F(N,T) = -kT \ln Z^N = \boxed{-NkT \ln\left(\frac{kT}{\hbar\omega}\right)}$$

$$S(N,T) = -\left(\frac{\partial F}{\partial T}\right)_N = = Nk \ln\left(\frac{kT}{\hbar\omega}\right) + Nk = \boxed{Nk\left[\ln\left(\frac{kT}{\hbar\omega}\right) + 1\right]}$$

II-10)

(a) $$Z = \sum_{l=0}^{\infty} (2l+1) \exp\left(-\frac{\hbar^2}{2IkT} l(l+1)\right)$$

with $$\theta_r = \frac{\hbar^2}{2Ik}$$

$$Z = \sum_{l=0}^{\infty} (2l+1) \exp\left(-\theta_r/T \; l(l+1)\right)$$

(b) high temperatures $T \gg \theta_r$

$$Z = \int_0^{\infty} (2l+1) \exp\left(-\theta_r/T \; l(l+1)\right) dl$$

$$x = \frac{\theta_r}{T} l(l+1)$$

$$dx = \frac{\theta_r}{T} (2l+1)\, dl$$

$$Z = \frac{T}{\theta_r} \int_0^{\infty} e^{-x} dx = \frac{T}{\theta_r} \left(-e^{-x}\right)_0^{\infty}$$

$$Z = \frac{T}{\theta_r} \qquad \ln Z = \ln T - \ln \theta_r$$

(c) $$\langle E \rangle = kT^2 \frac{\partial}{\partial T} \ln Z = kT^2 \cdot \frac{1}{T} = kT$$

$$U = N\langle E \rangle = NkT = \nu RT$$

$$C_V = \left(\frac{\partial U}{\partial T}\right)_V \Rightarrow C_V = \nu R$$