Physics PhD Qualifying Examination
Part I – Wednesday, January 19, 2011

Name:____
(please print)
Identification Number:________

STUDENT: Designate the problem numbers that you are handing in for grading in the appropriate left hand boxes below. Initial the right hand box.
PROCTOR: Check off the right hand boxes corresponding to the problems received from each student. Initial in the right hand box.

	1	
	2	
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	6	
	7	
	8	
	9	
	10	

Student's initials

problems handed in:

Proctor's initials

INSTRUCTIONS FOR SUBMITTING ANSWER SHEETS

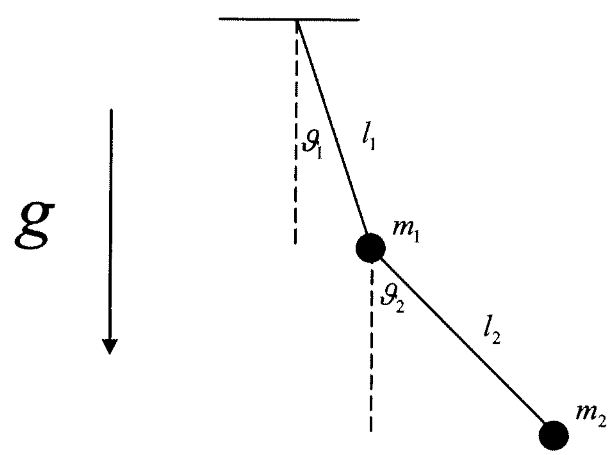
1. DO NOT PUT YOUR NAME ON ANY ANSWER SHEET. EXAMS WILL BE COLLATED AND GRADED BY THE ID NUMBER ABOVE.
2. Use at least one separate preprinted answer sheet for each problem. Write on only one side of each answer sheet.
3. Write your **identification number** listed above, in the appropriate box on each preprinted answer sheet.
4. Write the **problem number** in the appropriate box of each preprinted answer sheet. If you use more than one page for an answer, then number the answer sheets with both problem number and page (e.g. Problem 9 – Page 1 of 3).
5. Staple together all the pages pertaining to a given problem. Use a paper clip to group together all eight problems that you are handing in.
6. Hand in a total of *eight* problems. A passing distribution will normally include at least three passed problems from problems 1-5 (Mechanics) and three problems from problems 6-10 (Electricity and Magnetism). **DO NOT HAND IN MORE THAN EIGHT PROBLEMS.**
7. **YOU MUST SHOW ALL YOUR WORK.**

[I-1] [10]

Consider an object that is coasting horizontally subject to a drag force $f(v) = -bv - cv^2$, where v is the velocity and b and c are positive constants. Solve for v(t), that is, the velocity as a function of time, given an initial velocity v_0.

[I-2] [10]

Write down the Lagrangian of a double pendulum consisting of two particles of mass m_1 and m_2 connected by *massless rods* of length l_1 and l_2. Consider the motion to be in the *x-y* plane, with the gravitational force pointing along the *-y* axis. Write down the equation of motions. *You do not need to solve the equation of motions.*



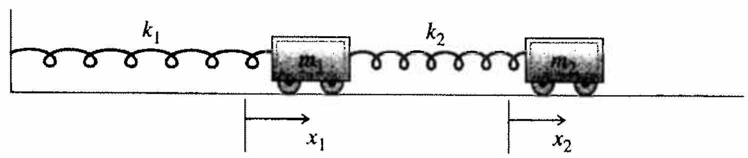
[I-3] [10]

Two carts of mass m_1 and m_2 are connected by a spring with spring constant k_2. One of the carts is connected to the wall by another spring with spring constant k_1. The only horizontal force acting on the two carts is the spring force. Consider the case of horizontal oscillations of the carts and

(a) Calculate the normal frequencies of the two carts for the case $m_1 = m_2 = m$ and k_1=3k and k_2=2k.

(b) Calculate the amplitudes a_{jr} of the normal modes $q_{jr} = a_{jr} \exp[i(\omega_r t - \delta_r)]$.

(c) Find the general solution for the coupled oscillations of the two carts.



[I-4] [10]

Two point-like objects, each with mass m, are connected by a *massless* rope of length l. The objects are suspended vertically near the surface of the Earth, so that one object is hanging below the other. Then the objects are released.

What is the tension in the rope? In obtaining your final result, assume that $l/R \ll 1$, and obtain your final answer in leading order of l/R

(The mass of the Earth is M, the radius of the Earth is R, the gravitational constant is G.)

[I-5] [4,4,2] (Multiple-Choice Questions)
(No partial credit; Full credit for correct answer per question, no details are needed to be shown; Must get two out of three questions correct to pass this problem)
Must CIRCLE the correct answers (i.e., not cross, check, shade, etc., but CIRCLE)

***(i)* [4pts.]** Two events occur on the x axis separated in time by Δt and in space by Δx. A reference frame, traveling at a speed less than the speed of light, in which the two events occur at the same place (same spatial coordinates):

(A) exists no matter what the values Δx and Δt.

(B) exists only if $|\Delta x / \Delta t| < c$.

(C) exists only if $|\Delta x / \Delta t| = c$.

(D) exists only if $|\Delta x / \Delta t| > c$.

(E) does not exist under any condition.

***(ii)* [4pts.]** Two events occur on the x axis separated in time by Δt and in space by Δx. A reference frame, traveling at a speed less than the speed of light, in which the two events occur at the same time:

(A) exists no matter what the values Δx and Δt.

(B) exists only if $|\Delta x / \Delta t| < c$.

(C) exists only if $|\Delta x / \Delta t| = c$.

(D) exists only if $|\Delta x / \Delta t| > c$.

(E) does not exist under any condition.

***(iii)* [2pts.]** A meson when at rest decays $10\mu s$ after it is created. If moving in the laboratory at $0.98c$, its lifetime according to laboratory clock would approximately be:

(A) the same.

(B) $20.52\mu s$.

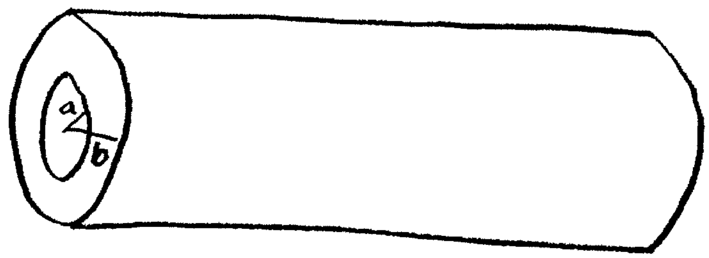
(C) $2.00\mu s$.

(D) $50.25\mu s$.

(E) none of the above.

[I-6] [10]

A capacitor is made from two *infinitely long* conductors with coaxial cylindrical surfaces as shown below. The inner conductor, with radius a, is solid, the outer conductor, with radius b, is a cylindrical shell, with vacuum in between. Find the capacitance of a length L of this system.



[I-7] [10]

Write the microscopic Maxwell equations in differential form and then show that the electric and magnetic fields, $\boldsymbol{E}$ and $\boldsymbol{B}$, satisfy the homogeneous wave equation, when there are no external sources. Show that the solutions are transverse waves.

[I-8]   [5,5]

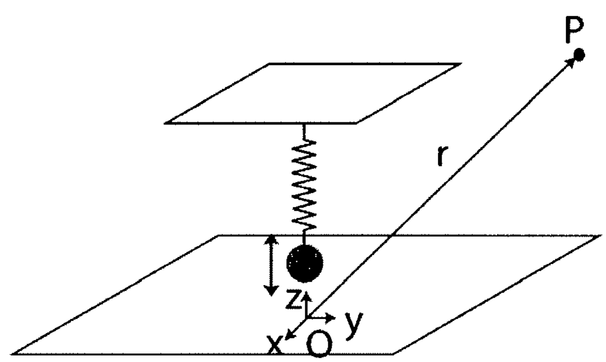
A long straight wire of radius b carries a current I in response to a voltage V between the ends of the wire.

(a) Calculate the Poynting vector $\vec{S}$ inside the wire ($r \le b$) for this DC voltage.
(b) Obtain the energy flux per unit length at the surface of the wire. Compare this result with Joule heating of the wire and comment on the physical significance.

[I-9]   [5,5]

A particle with charge q and mass m is hanging from the ceiling by a spring. A large and flat perfect conductor is placed under the particle. The particle is displaced downward, and released. The particle starts oscillating slowly between z=0 and z=d/2 with angular frequency ω. (Assume the charge stops just above the plate when it comes to z=0, and does not come in contact with the plate.)

Calculate
(a) the scalar and
(b) the vector potential
at point P shown in the figure below, using the *far field approximation.* The origin of the coordinate system, O, is on the plate and the z-direction is vertical. r is the distance between O and P. The spring is massless, and assume $\omega d/c$<<1. (Hint: assume the image principle can be used.)



[I-10] **[3,3,4]**

A charge e is moving at a constant speed, V along the x-axis of coordinate system K. Coordinate system K' is moving together with the charge. The x, y, z axes of coordinate system K are parallel to the x', y', z' axes in K' coordinate system, respectively.
(a) Express the x, y, z components of electric field in K using x, y, and z.
(b) Find the electric field in the direction with angle θ away from the direction of the velocity.
(c) What are the directions in which the electric filed becomes maximum and minimum?

Recall that the Lorentz transformation of the electromagnetic field is give by

$$\boldsymbol{E}_{\parallel} = \boldsymbol{E}'_{\parallel}, \qquad \boldsymbol{E}_{\perp} = \gamma(\boldsymbol{E}'_{\perp} - \boldsymbol{V} \times \boldsymbol{B}'),$$

where $\gamma = \left(1 - V^2/c^2\right)^{-1/2}$, c is speed of light, and $\parallel$ and $\perp$ represent the parallel and perpendicular components relative to the direction of $\boldsymbol{V}$, respectively.

Solutions

I-1

$$m\frac{dv}{dt} = -bv - cv^2$$

$$\int_{v_0}^{v} \frac{dv}{v\left(1+\frac{c}{b}v\right)} = -\int_0^t \frac{b}{m}\,dt = -\frac{b}{m}t$$

$$= -\ln \frac{1+\frac{c}{b}v}{v}\Bigg|_{v_0}^{v} = -\ln\left(\frac{v_0}{1+\frac{c}{b}v_0}\cdot\frac{1+\frac{c}{b}v}{v}\right)$$

$$\frac{v_0}{1+\frac{c}{b}v_0}\cdot\frac{1+\frac{c}{b}v}{v} = e^{bt/m}$$

$$\frac{v_0}{1+\frac{c}{b}v_0}\left(1+\frac{c}{b}v\right) = v\,e^{bt/m}$$

$$v\,e^{bt/m} - \frac{v_0}{1+\frac{c}{b}v_0}\frac{c}{b}v = \frac{v_0}{1+\frac{c}{b}v_0}$$

$$v = \frac{v_0}{1+\frac{c}{b}v_0}\left[e^{bt/m} - \frac{c}{b}\frac{v_0}{1+\frac{c}{b}v_0}\right]^{-1}$$

I-2

Problem 1.2 Write the Lagrangian of a double pendulum consisting of two particles of mass m_1 and m_2 connected by massless rods of length l_1 and l_2. Consider the motion to be in the x-y plane, with the gravitational force along the -y axis. Write the equation of motions. **You do not need to solve the equation of motions.**

Solution:

For the first particle:

$$T_1 = \frac{1}{2}m_1 l_1^2 \dot{\theta}_1^2$$

$$V_1 = -m_1 g l_1 cos(\theta_1)$$

For the second particle:

$$x_2 = l_1 sin(\theta_1) + l_2 sin(\theta_2)$$

$$y_2 = l_1 cos(\theta_1) + l_2 cos(\theta_2)$$

The kinetic energy is

$$\begin{aligned} T_2 &= \frac{1}{2}m_2(\dot{x}_2^2 + \dot{y}_2^2) \\ &= \frac{1}{2}(l_1^2\dot{\theta}_1^2 + l_2^2\dot{\theta}_2^2 + 2l_1 l_2 cos(\theta_1 - \theta_2)\dot{\theta}_1\dot{\theta}_2 \end{aligned}$$

The potential energy is

$$V_2 = -m_2 g y_2 = -m_2 g(l_1 cos(\theta_1) + l_2 cos(\theta_2))$$

which gives the Lagrangian

$$\begin{aligned} \mathcal{L} = &\frac{1}{2}(m_1 + m_2)l_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2 l_2^2\dot{\theta}_2^2 + m_2 l_1 l_2 cos(\theta_1 - \theta_2)\dot{\theta}_1\dot{\theta}_2 \\ &+ (m_1 + m_2)g l_1 cos(\theta_1) + m_2 g l_2 cos(\theta_2) \end{aligned}$$

Equations of motion:

$$\frac{\partial \mathcal{L}}{\partial \theta_i} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}_i}\right)$$

for $i = 1, 2$.

$$\begin{aligned} &(m_1 + m_2)g l_1 \ddot{\theta}_1 + l_1 l_2 m_2(cos[\theta_1 - \theta_2]\ddot{\theta}_2 - sin[\theta_1 - \theta_2]\dot{\theta}_2)(\dot{\theta}_1 - \dot{\theta}_2) \\ &\quad = -(m_1 + m_2)g l_1 sin[\theta_1] - m_2 l_1 l_2 \dot{\theta}_1\dot{\theta}_2 sin[\theta_1 - \theta_2] \end{aligned}$$

$$m_2 g l_2^2 \ddot{\theta}_2 + l_1 l_2 m_2 (cos[\theta_1 - \theta_2]\ddot{\theta}_1 - sin[\theta_1 - \theta_2]\dot{\theta}_1)(\dot{\theta}_1 - \dot{\theta}_2)$$
$$= -m_2 g l_2 sin[\theta_2] + m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 sin[\theta_1 - \theta_2]$$

These are non-linear equations that cannot be solved easily.

$\boxed{I-3}$

$$m_1\ddot{x}_1 = -k_1x_1 - k_2(x_1-x_2)$$
$$m_2\ddot{x}_2 = -k_2(x_2-x_1)$$

$$m_1\ddot{x}_1 + k_1x_1 + k_2(x_1-x_2) = 0$$
$$m_2\ddot{x}_2 + k_2(x_2-x_1) = 0$$

$$x_1 = B_1 e^{i\omega t}$$
$$x_2 = B_2 e^{i\omega t}$$

$$-m_1 B_1\omega^2 + B_1(k_1+k_2) - B_2k_2 = 0$$
$$-m_2 B_2\omega^2 + k_2B_2 - B_1k_2 = 0$$

$$\begin{pmatrix} -m_1\omega^2 + (k_1+k_2) & -k_2 \\ -k_2 & -m_2\omega^2 + k_2 \end{pmatrix}\begin{pmatrix} B_1 \\ B_2 \end{pmatrix} = 0$$

$$-m_1\omega^2 + (k_1+k_2)(-m_2\omega^2 k_2) - k_2^2 = 0$$

$m_1 = m_2$

$k_1 = 3k$

$k_2 = 2k$

$$(-m\omega^2 + 5k)(-m\omega^2 + 2k) - 4k^2 = 0$$
$$m^2\omega^4 - 7km\omega^2 + 10k^2 - 4k^2 = 0$$
$$m^2\omega^4 - 7km\omega^2 + 6k^2 = 0$$
$$\omega^2 = \frac{7km}{2m^2} \pm \frac{\sqrt{49k^2m^2 - 24m^2k^2}}{2m^2}$$
$$= \frac{7km}{2m^2} \pm \frac{\sqrt{25m^2k^2}}{2m^2} = \frac{7km}{2m^2} \pm \frac{5mk}{2m^2}$$
$$\omega_1^2 = \frac{12k}{2m} \qquad \omega_2^2 = \frac{k}{m} \qquad \omega_1 = \sqrt{\frac{6k}{m}}, \quad \omega_2 = \sqrt{\frac{k}{m}}$$

eigenvalues

Find eigenvectors for eigenvalues ω_1, ω_2

$$\begin{bmatrix} k_1 + k_2 - m\omega^2 & -k_2 \\ -k_2 & k_2 - m_2\omega^2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = 0$$

for $\omega_1 = \sqrt{\frac{6k}{m}}$

$$\begin{bmatrix} 5k - m\left(\frac{6k}{m}\right) & -2k \\ -2k & 2k - m\left(\frac{6k}{m}\right) \end{bmatrix} \Rightarrow \begin{bmatrix} -k & -2k \\ -2k & -4k \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} =$$

$$a_1 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

for $\omega_2 = \sqrt{\frac{k}{m}}$

$$\begin{bmatrix} 5k - k & -2k \\ -2k & 2k - k \end{bmatrix} \Rightarrow \begin{bmatrix} 4k & -2k \\ -2k & k \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = 0 \quad a_2 =$$

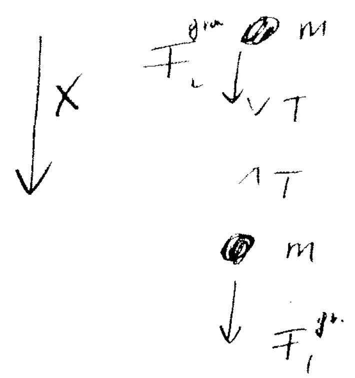
$$a_1 = \begin{bmatrix} 2 \\ -1 \end{bmatrix} \qquad a_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

general solution $\vec{q} = \sum_i c_i \vec{a} \cos(\omega_r t - \delta_r)$

$$\vec{q} = c_1 \begin{bmatrix} 2 \\ -1 \end{bmatrix} \cos\left(\sqrt{\frac{6k}{m}}\, t - \delta\right) + c_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} \cos\left(\sqrt{\frac{k}{m}}\, t - \right.$$

I-4 [10]

Two points like objects, each with mass m, are connected by a *massless* rope of length l. The objects are suspended vertically near the surface of the Earth, so that one object is hanging below the other. Then the objects are released. What is the tension in the rope? (The mass of the Earth is M, the radius of the Earth is R, the gravitational constant is G.)



$$\begin{cases} m\ddot{x}_1 = F_1^{gr.} - T \\ m\ddot{x}_2 = F_2^{gr.} + T \end{cases}$$

$$\ddot{x}_1 = \ddot{x}_2$$

$$F_1^{gr} - T = F_2^{gr} + T$$

$$G\frac{mM}{R^2} - T = G\frac{mM}{(R+l)^2} + T$$

$$2T = GmM\left\{\frac{1}{R^2} - \frac{1}{(R+l)^2}\right\}$$

$$T = G\frac{mM}{2}\left\{\frac{1}{R^2} - \frac{1}{R^2(1+l/R)^2}\right\} = G\frac{mM}{2R^2}\{\underbrace{1-(1+l/R)^{-2}}_{1-(1-2l/R)}\}$$

$$\simeq G\frac{mM}{2R^2}\left\{1 - 1 + 2\frac{l}{R}\right\} = G\frac{mM}{R^2}\cdot\frac{l}{R}$$

1-5 [4,4,2] (No partial credit; Full credit for correct answer per question, no details are needed to be shown)

Must CIRCLE the correct answers (i.e., not cross, check, shade, etc., but CIRCLE)

(a) [4pts.] Two events occur on the x axis separated in time by Δt and in space by Δx. A reference frame, traveling at a speed less than the speed of light, in which the two events occur at the same place (same spatial coordinates):

(A) exists no matter what the values Δx and Δt.

(B) exists only if $|\Delta x/\Delta t| < c$.

(C) exists only if $|\Delta x/\Delta t| = c$.

(D) exists only if $|\Delta x/\Delta t| > c$.

(E) does not exist under any condition.

$0 = \Delta x' = \gamma(\Delta x - u\Delta t)$

$\left|\frac{\Delta x}{\Delta t}\right| = |u| < c$

("time-like" interval)

(b) [4pts.] Two events occur on the x axis separated in time by Δt and in space by Δx. A reference frame, traveling at a speed less than the speed of light, in which the two events occur at the same time:

(A) exists no matter what the values Δx and Δt.

(B) exists only if $|\Delta x/\Delta t| < c$.

(C) exists only if $|\Delta x/\Delta t| = c$.

(D) exists only if $|\Delta x/\Delta t| > c$.

(E) does not exist under any condition.

$0 = \Delta t' = \gamma\left(-\frac{u}{c^2}\Delta x + \Delta t\right)$

$\left|\frac{\Delta x}{\Delta t}\right| = \frac{c^2}{|u|} = c \cdot \underbrace{\frac{c}{|u|}}_{>1} > c$

("space-like" interval)

(c) [2pts.] A meson when at rest decays $10\mu s$ after it is created. If moving in the laboratory at $0.98c$, its lifetime according to laboratory clock would approximately be:

(A) the same.

(B) $20.52\mu s$.

(C) $2.00\mu s$.

(D) $50.25\mu s$.

(E) none of the above.

$\tau = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \cdot \tau_0$

I-6

Using Gauss' Law and linear charge density λ:

$$2\pi r E(r) h = \frac{1}{\epsilon_0}\lambda h \Rightarrow E = \frac{\lambda}{2\pi\epsilon_0 r}$$

Then

$$V = \int_a^b dr\, E(r) = \frac{\lambda}{2\pi\epsilon_0}\ln\frac{b}{a}$$

$$C = \frac{Q}{V} = \frac{\lambda L}{\frac{\lambda}{2\pi\epsilon_0}\ln\frac{b}{a}} = \frac{2\pi\epsilon_0 L}{\ln\left(\frac{b}{a}\right)}$$

I - 7

Problem 1.7 Write microscopic Maxwell equations in differential form and then show that the electric and magnetic field satisfy the homogeneous wave equation, when there are no external sources. Show that the solution are transverse waves.

Solution:

$$\nabla \cdot E = \frac{\rho}{\epsilon_0}$$
$$\nabla \cdot B = 0$$

are the Gauss laws.

$$\nabla \times E + \frac{\partial B}{\partial t} = 0$$

is the Faraday's law.

$$\nabla \times B - \mu_0\epsilon_0\frac{\partial E}{\partial t} = \mu_0 J$$

where

$$\mu_0\epsilon_0 = \frac{1}{c^2}$$

is Ampere- Maxwell law.

Wave equation:

Use the identity

$$\nabla \times \nabla \times E = -\nabla^2 E + \nabla(\nabla \cdot E)$$

and the same for B. For simplicity, assume $J = 0$, and $\rho = 0$ (no sources).

Take the cross product on Faraday's law to get

$$\nabla \times \nabla \times E + \frac{\partial \nabla \times B}{\partial t} = 0$$

and the identify before, Gauss, and Amperes-Maxwell law to get

$$\nabla \times \nabla \times E = -\nabla^2 E + \nabla(\nabla \cdot E) + \mu_0\epsilon_0\frac{\partial^2 E}{\partial t^2} = 0$$
$$\nabla^2 E - \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2} = 0$$

Same works for B.

Now assume a plane wave solution

$$E = E_0 e^{ik\cdot x - wt}$$

The Maxwell equations give

$$k \cdot E = 0$$
$$k \cdot B = 0$$

That is, the wave propagate perpendicular to the fields. From Faradays,

$$k \times E = wB$$

shows that E is perpendicular to B (for real $|k|$ and w and $\neq 0$).

(I-8) Solution:

(a) Let us calculate the flux of the Poynting vector. Introduce cylindrical coordinates with unit vectors $\hat{e}_\rho$, $\hat{e}_\theta$ and $\hat{z}$. Current flows along the wire in the z-direction and the electric field $\vec{E} = E\hat{z}$. Using one of the Maxwell's equation in vacuum the fact that conditions are stationary and Stokes' theorem

$$\vec{\nabla}\times\vec{B} = \frac{4\pi}{c}\vec{J} + \frac{1}{c}\frac{\partial E}{\partial t}$$

$$\int_A \vec{\nabla}\times\vec{B}\cdot d\vec{A} = \oint_c \vec{B}\cdot d\vec{l} = \frac{4\pi}{c}\int_A \vec{J}\cdot d\vec{A}$$

here $\vec{J}$ is the current density and $\vec{A}$ the surface. At any given radius r, B_θ is constant, so we have

$$2\pi r B = \frac{4\pi}{c} J \pi r^2, \quad \vec{B} = \frac{2\pi}{c} J r \hat{e}_\theta$$

$$\vec{S} = \frac{c}{4\pi}\vec{E}\times\vec{B} = \frac{c}{4\pi}\frac{2\pi}{c} J r E(\hat{z}\times\hat{e}_\theta)$$

$$\vec{S} = \frac{1}{2} J r E\,\hat{e}_\rho$$

Using the relation between current density and total current

$$J = I/(\pi b^2):$$

(I-8) continued.

$$\vec{S} = -\frac{I E r}{2\pi b^2}\,\hat{e}_\rho\ , \quad \vec{S}(b) = -\frac{I E}{2\pi b}\,\hat{e}_\rho$$

(b) The Poynting flux per unit length is then $\vec{S}\cdot 2\pi \vec{b} = -IE$. So the flux enters the wire, and we see that the dissipated power per unit length IE is equal to the total incoming S-flux in agreement with Poynting's theorem

$$\frac{\partial u}{\partial t} = -\vec{J}\cdot\vec{E} - \vec{\nabla}\cdot\vec{S}$$

where u is the energy density. Under stationary conditions as here $\frac{\partial u}{\partial t}=0$, and we have

$$\int_V \vec{J}\cdot\vec{E}\,d^3x = -\int_V \vec{\nabla}\cdot\vec{S}\,dV = -\int_A \vec{S}\cdot d\vec{A} = IE.$$

I - 9

Ans.

The charge is oscillating between z=0 and d/2 at angular frequency ω. The position of charge q is given by follow.

$$z_+(t) = \frac{d}{2}\cos\omega t$$

Using the principle of images, the effect of the conductor plate is same as $-q$ charge existing at $-z_+$ position. This system is equivalent to an oscillating dipole moment, $p(t) = qd\cos\omega t$.

Scalar potential at P:

Define the distance from the charge to point P as r_+, and the distance from the image charge to the point P as r_-.

Scalar potential is given by follow.

$$V(r,t) = \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{r_+} - \frac{q}{r_-}\right] = \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{r_+} - \frac{1}{r_-}\right]$$

Since d<<r,

$$r_\pm = \sqrt{r^2 \mp r z_\pm \cos\theta + \left(\frac{d}{2}\right)^2} \cong r\left(1 \mp \frac{z_\pm}{2r}\cos\theta\right) \text{ and } \frac{1}{r_\pm} \cong \frac{1}{r}\left(1 \pm \frac{z_\pm}{2r}\cos\theta\right)$$

Also, ωd/c<<1

$$\begin{aligned} z_+(t - r_{+/-}/c) &= \frac{d}{2}\cos\left[\omega(t - r_{+/-}/c)\right] = \frac{d}{2}\cos\left[\omega(t - r/c) \mp \frac{\omega z_\pm}{2c}\cos\theta\right] \\ &= \frac{d}{2}\cos\left[\omega(t - r/c)\right]\cos\left[\frac{\omega z_\pm}{2c}\cos\theta\right] \pm \frac{d}{2}\sin\left[\omega(t - r/c)\right]\sin\left[\frac{\omega z_\pm}{2c}\cos\theta\right] \\ &\cong \frac{d}{2}\cos\left[\omega(t - r/c)\right] \end{aligned}$$

$$\begin{aligned} V(r,t) &= \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{r_+} - \frac{1}{r_-}\right] \cong \frac{q}{4\pi\varepsilon_0 r}\left[\frac{z_+(t - r_+/c)}{2r} + \frac{z_-(t - r_-/c)}{2r}\right]\cos\theta \\ &\cong \frac{qd\cos\theta}{4\pi\varepsilon_0 r^2}\cos\left[\omega(t - r/c)\right] \end{aligned}$$

Vector potential at P:

Current density: $\mathbf{j}(t) = \sum q\mathbf{v}\hat{z} = -q\frac{d}{2}\omega\sin\omega t\hat{z} - (-q)\left(-\frac{d}{2}\omega\sin\omega t\hat{z}\right) = -qd\omega\sin\omega t\hat{z}$

$$\mathbf{A}(\mathbf{x},t)=\frac{\mu_0}{4\pi}\int_{-d/2}^{d/2}\frac{-qd\omega\sin\left[\omega(t-r/c)\right]}{r}dz$$

The integration introduces factor of d (<<r). In the first order approximation, we can replace r with the value at the center.

$$\rightarrow \quad \mathbf{A}(\mathbf{x},t)=-\frac{\mu_0 qd\omega}{4\pi}\frac{\sin\left[\omega(t-r/c)\right]}{r}\hat{\mathbf{z}}$$

[I -10]

(a)

$$E_x = E_x' = \frac{ex'}{R'^3}$$

$$E_y = \frac{E_y'}{\sqrt{1-\frac{V^2}{c^2}}} = \frac{ey'}{R'^3\sqrt{1-\frac{V^2}{c^2}}}$$

$$E_z = \frac{ez'}{R'^3\sqrt{1-\frac{V^2}{c^2}}}$$

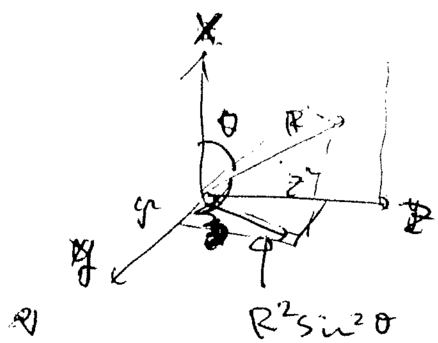
~~Fro~~ using Lorentz transform

$$\Rightarrow R^{*2} = (x-Vt)^2 + \left(1-\frac{V^2}{c^2}\right)(\ldots$$

$$\Rightarrow \quad \mathbb{E} = \left(1-\frac{V^2}{c^2}\right)\frac{e\mathbb{R}}{R^{*2}}$$

(b)

$$y^2 + z^2 = R^2\sin^2\theta$$



$$\Rightarrow R^{*2} = R^2\left(1-\frac{V^2}{c^2}\sin^2\theta\right)$$

$$\Rightarrow \quad \mathbb{E} = \frac{e\mathbb{R}}{R^3}\,\frac{1-\frac{V^2}{c^2}}{\left(1-\frac{V^2}{c^2}\sin^2\theta\right)^{3/2}}$$

$$\theta = 0 \qquad \mathbb{E}_\parallel = \frac{e}{R^2}\left(1-\frac{V^2}{c^2}\right) \qquad \text{(along } x\text{)}$$

$$\theta = \pi \qquad \mathbb{E}_\perp = \frac{e}{R^2}\,\frac{1}{\sqrt{1-\frac{V^2}{c^2}}} \qquad \text{(perpendicular to } x\text{)}$$

Physics PhD Qualifying Examination
Part II – Friday, January 21 2011

Name:____
(please print)
Identification Number:________

STUDENT: insert a check mark in the left boxes to designate the problem numbers that you are handing in for grading.
PROCTOR: check off the right hand boxes corresponding to the problems received from each student. Initial in the right hand box.

	1	
	2	
	3	
	4	
	5	
	6	
	7	
	8	
	9	
	10	

Student's initials

problems handed in:

Proctor's initials

INSTRUCTIONS FOR SUBMITTING ANSWER SHEETS

1. DO NOT PUT YOUR NAME ON ANY ANSWER SHEET. EXAMS WILL BE COLLATED AND GRADED BY THE ID NUMBER ABOVE.
2. Use at least one separate preprinted answer sheet for each problem. Write on only one side of each answer sheet.
3. Write your **identification number** listed above, in the appropriate box on the preprinted sheets.
4. Write the **problem number** in the appropriate box of each preprinted answer sheet. If you use more than one page for an answer, then number the answer sheets with both problem number and page (e.g. Problem 9 – Page 1 of 3).
5. Staple together all the pages pertaining to a given problem. Use a paper clip to group together all eight problems that you are handing in.
6. Hand in a total of *eight* problems. A passing distribution will normally include at least four passed problems from problems 1-6 (Quantum Physics) and two problems from problems 7-10 (Thermodynamics and Statistical Mechanics). **DO NOT HAND IN MORE THAN EIGHT PROBLEMS.**
7. **YOU MUST SHOW ALL YOUR WORK.**

[II-1] **[10]**

A particle is in a one-dimensional potential well given by $V(x) = -c\delta(x)$, where $\delta(x)$ is the Dirac delta function and $c > 0$ is a constant.
Find the energy and the *normalized* wave-function of the *bound state(s).*

Hint: You must carefully consider and study the possible discontinuity in the derivative of the wave function $\psi'(x)$ at $x = 0$. You can do this by integrating Schrödinger's equation for the above system from $-\varepsilon$ to $+\varepsilon$ and the let $\varepsilon \to 0$.
This is at the heart of this problem, and without a meaningful treatment and analysis of this discontinuity, you will *not* pass this problem.

[II-2] **[10]**

Consider a particle of mass m in a one-dimensional box with infinitely high walls at x=0 and x=L.

(a) Find the eigenenergies E_n and normalized eigenfunctions ϕ_n for the particle in the box ($n = 1,2,\ldots$).
(b) Calculate the first order correction to the unperturbed energy $E_3{}^{(0)}$ for the particle due to the following perturbation $H' = 10^{-3}\, E_1 \dfrac{x}{L}$. Here, $E_1 = \dfrac{\hbar^2\pi^2}{2mL^2}$ is a constant.

[II-3] **[10]**

Consider an electron spin and an arbitrary direction defined by the unit vector $\vec{e} = (\sin\vartheta\cos\varphi, \sin\vartheta\sin\varphi, \cos\vartheta)$ in the three-dimensional space, specified by the polar (ϑ) and the azimuth (φ) angles. In the usual representation, the electron spin operator can be expressed in terms of the Pauli matrices

$$\vec{S} = \frac{\hbar}{2}\vec{\sigma}, \qquad \vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z),$$

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \ \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Now assume that we measure S_z, and it is $+\hbar/2$.

What is the probability that the component of the spin along the direction $\vec{e}$ is $+\hbar/2$?

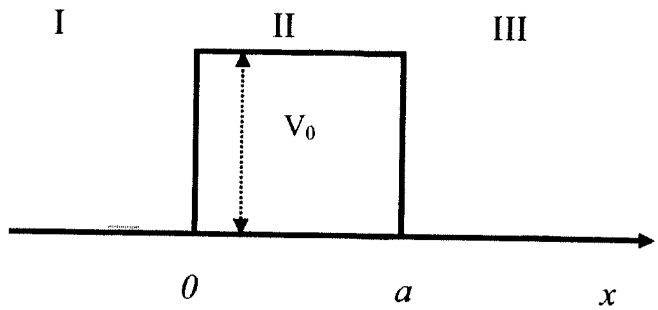
[II-4] **[10]**

A particle with mass m is scattered by a Yukawa-potential, $V(r) = \dfrac{V_0}{\alpha r} e^{-\alpha r}$.

Using the Born approximation, find the differential scattering cross section and then calculate total scattering cross section.

[II-5] **[10]**

Consider a truly monoenergetic beam of electrons of energy E. The beam of electrons is incident from the left onto a rectangular potential energy barrier with height $V_0 > 0$ and width a. (see Fig. below).



(a) Write down the Hamiltonian H for the electrons in region I, II and III.

(b) Find the eigenfunctions $\Psi(x)$ in regions I ($x < 0$), II ($0 \le x \le a$) and III ($x > a$) in the case when the energy of the electron is less than the height of the barrier, i.e., $0 < E < V_0$.

(c) The eigenfunctions assume the form $\psi(x) = C\exp(ikx)$ for $x \to \infty$ and $\psi(x) = A\exp(ikx) + B\exp(-ikx)$ for $x \to -\infty$.

Calculate the transmission coefficient $T = \left|\frac{C}{A}\right|^2$.

[II-6] **[10]**

A harmonic oscillator of frequency ω is in its ground state (n=0) in the infinite past. It is subjected to a time-dependent perturbation $V_{\text{pert}}(x) = \lambda\, x \exp(-\alpha t^2)$, where x is the position of the particle relative to the equilibrium point. To leading order in λ, determine the probability that the system will be found in the first excited state (n=1) in the infinite future.

[II-7] **[10]**

Two blocks are made of identical metal of specific heat (per unit mass) c and are isolated from the rest of the outside world, but not from each other. Initially, the first block, with mass $2m$, has temperature T; the second block, with mass m, has temperature $4T$. They are then brought into contact with each other, and eventually they come to thermal equilibrium with each other.

What is the total change in entropy ΔS_{total} for the system (the two blocks combined)? You must express your answer in terms of m, c, and T (not all of them may show up in your final answer).

[II-8] **[10]**

Consider the Van der Waals gas given by the equation of state and energy given by

$$\left(P + \frac{a}{v^2}\right)(v - b) = RT \quad ,$$

$$u(T, v) = c_v T - \frac{a}{v} \quad ,$$

where a, b, R, and c_v are positive constants. The gas undergoes a reversible (quasi-static) adiabatic expansion from v to $2v$. Obtain the ratio of the final and initial temperatures, T_2 / T_1. You must express your answer in terms of the variables and constants given above (i.e., do not reinvent new constant or redefine new variables)

[II-9] **[10]**

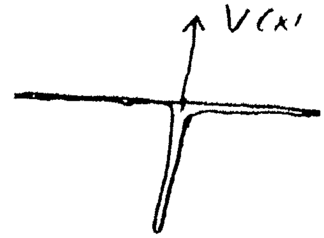
The partition function of a classical one-dimensional system can be written as $Z = \int dp\, dq \exp(-E(p,q)/kT)$, where p is momentum, q is position, and $E(p,q)$ is the classical energy. For the simple harmonic oscillator of frequency ω and mass m, determine the entropy.

[II-10] **[10]**

Solve for the pressure of a neutron gas at $T = 0$ (neutrons are spin ½ particles). How does it compare to the pressure of an ideal gas under similar N; T; V conditions?

Solutions

II-1 $V(x) = -c\,\delta(x)$ $\quad (c > 0)$



(1) $$-\frac{\hbar^2}{2m}\psi''(x) + V(x)\psi(x) = E\psi(x)$$

$$x \neq 0: \quad -\frac{\hbar^2}{2m}\psi''(x) = E\psi(x)$$

$$\psi''(x) = -\frac{2mE}{\hbar^2}\psi(x)$$

for bound state one needs $\psi(x) \xrightarrow[x\to\pm\infty]{} 0$.
thus $E < \infty$

(2) $$\psi(x) = \begin{cases} A e^{\kappa x} & x < 0 \\ B e^{-\kappa x} & x > 0 \end{cases}$$

where $\kappa = \sqrt{\dfrac{2m|E|}{\hbar^2}}$

<u>The heart of this problem is to determine the energy E of the bound state(s)</u>

Integrating (1) from $-\varepsilon$ to $+\varepsilon$ and take $\varepsilon \to 0$:

$$\psi''(x) = -\frac{2m}{\hbar^2}[E - V(x)]\,\psi(x)$$

$$\int_{-\varepsilon}^{+\varepsilon}\psi''(x)\,dx = -\frac{2m}{\hbar^2}\left[E\int_{-\varepsilon}^{+\varepsilon}\psi(x)\,dx - \int_{-\varepsilon}^{+\varepsilon}V(x)\psi(x)\,dx\right.$$

$\psi(x)$ must be continuous everywhere (including at $x = 0$)

$\Rightarrow \quad A = B$

$\varepsilon \to 0$ $$\psi'(x)\Big|_{-\varepsilon}^{+\varepsilon} = -\frac{2m}{\hbar^2}\left[\cancel{0} - \int_{-\varepsilon}^{+\varepsilon}(-c\,\delta(x))\,\psi(x)\,dx\right]$$

$$\psi'(+0) - \psi'(-0) = -\frac{2mc}{\hbar^2}\psi(0)$$

From (2):

$$-\kappa A - \kappa A = -\frac{2mc}{\hbar^2}A$$

$$2\kappa \cancel{A} = \frac{2mc}{\hbar^2}\cancel{A}$$

$$\Rightarrow \quad \boxed{\kappa = \frac{mc}{\hbar^2}}$$

and

$$|E| = \frac{\hbar^2\kappa^2}{2m} = \frac{\hbar^2}{2m}\,\frac{m^2c^2}{\hbar^4} = \frac{mc^2}{2\hbar^2}$$

So, $\boxed{E = -\frac{mc^2}{2\hbar^2}}$ a single bound state

From normalization:

$$\psi(x) = A e^{-\kappa|x|} \qquad \int_{-\infty}^{+\infty}|\psi(x)|^2dx = 1 \quad \Rightarrow \psi(x) = \sqrt{\kappa}\,e^{-\kappa|x|}$$

$$\boxed{\psi(x) = \sqrt{\frac{mc}{\hbar^2}}\; e^{-\frac{mc}{\hbar^2}|x|}}$$

II-2 Perturbation theory / time independent

(a) eigenenergies $E_n = n^2 E_1$ with $E_1 = \frac{\hbar^2 \pi^2}{2mL^2}$

eigenfunctions $\phi_n = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right)$

(b) first-order correction to $E_3^{(0)}$

$$E_3^{(1)} = \langle \phi_3 | H' | \phi_3 \rangle$$

$$E_3^{(1)} = \frac{2}{L^2} 10^{-3} E_1 \int_0^L x \sin^2\left(\frac{3\pi x}{L}\right) dx$$

$y = \frac{3\pi x}{L}$ $\qquad$ $x = \frac{L}{3\pi} y$ $\qquad$ $dx = \frac{L}{3\pi} dy$

$$= \frac{2}{L^2} 10^{-3} E_1 \left(\frac{L}{3\pi}\right)^2 \int_0^{3\pi} y \sin^2 y \, dy$$

$$= \frac{2}{9\pi^2} 10^{-3} E_1 \int_0^{3\pi} y \sin^2 y \, dy \quad *$$

$$= \frac{2}{9\pi^2} 10^{-3} E_1 \left[\frac{y^2}{4} - \frac{y \sin 2y}{4} - \frac{\cos 2y}{8}\right]_0^{3\pi}$$

$$= \frac{2}{9\pi^2} 10^{-3} E_1 \left[\frac{9\pi^2}{4} - \frac{3\pi \sin 6\pi}{4} - \frac{\cos 6\pi}{8} - \left[0 - 0 - \frac{\cos 0}{8}\right]\right]$$

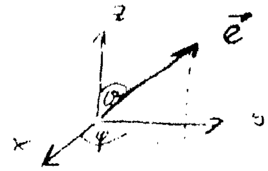
where $\sin 6\pi = 0$.

$$= \frac{2}{9\pi^2} 10^{-3} E_1 \left[\frac{9\pi^2}{4} - \frac{1}{8} - \left(-\frac{1}{8}\right)\right]$$

$$= \frac{2}{9\pi^2} 10^{-3} E_1 \frac{9\pi^2}{4}$$

$$E_3^{(1)} = 10^{-3} \frac{E_1}{2}$$

* integral 17.17.10 Schaum Outline, Mathematical Handbook of Formulas and Tables, 2nd ed.



$\boxed{\text{II} \cdot 3}$

$$\sigma_e = \vec{e} \cdot \vec{\sigma} = \sin\theta \cos\varphi\, \sigma_x + \sin\theta \sin\varphi\, \sigma_y + \cos\theta\, \sigma_z$$

$$= \sin\theta\cos\varphi \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \sin\theta \sin\varphi \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \cos\theta \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & \sin\theta\cos\varphi \\ \sin\theta\cos\varphi & 0 \end{pmatrix} + \begin{pmatrix} 0 & -i\sin\theta\sin\varphi \\ i\sin\theta\sin\varphi & 0 \end{pmatrix} + \begin{pmatrix} \cos\theta & 0 \\ 0 & -\cos\theta \end{pmatrix}$$

$$= \begin{pmatrix} \cos\theta & \sin\theta\, e^{-i\varphi} \\ \sin\theta\, e^{i\varphi} & -\cos\theta \end{pmatrix}$$

eigenvalues & eigenvectors: $\sigma_e |s_i\rangle = \lambda_i |s_i\rangle \quad i=1,2$

$$\begin{vmatrix} \cos\theta - \lambda & \sin\theta\, e^{-i\varphi} \\ \sin\theta\, e^{i\varphi} & -\cos\theta - \lambda \end{vmatrix} = (\lambda - \cos\theta)(\lambda + \cos\theta) - \sin^2\theta = 0$$

$$\lambda^2 - \cos^2\theta - \sin^2\theta = 0$$

$\lambda^2 = 1 \qquad \boxed{\lambda = \pm 1}$ (regardless of the direction $\vec{e}$)

$\underline{\lambda = 1:}$ $(\cos\theta - 1)\, a_1 + \sin\theta\, e^{-i\varphi}\, a_2 = 0$

$$a_2 = \frac{1 - \cos\theta}{\sin\theta}\, e^{i\varphi} a_1 = \frac{2\sin^2(\theta/2)}{2\sin(\theta/2)\cos(\theta/2)}\, e^{i\varphi} a_1 = \frac{\sin(\theta/2)}{\cos(\theta/2)}\, e^{i\varphi} a_1$$

normalized eigenvector:

$$|s_1\rangle = \begin{pmatrix} \cos(\theta/2) \\ \sin(\theta/2)\, e^{i\varphi} \end{pmatrix}$$

$\underline{\lambda = -1:)}$ $(\cos\theta + 1)\, a_1 + \sin\theta\, e^{-i\varphi} a_2 = 0$

$$a_2 = -\frac{1+\cos\theta}{\sin\theta} e^{i\varphi} a_1 = -\frac{2\cos^2(\theta/2)}{2\sin(\theta/2)\cos(\theta/2)} e^{i\varphi} a_1 = -\frac{\cos(\theta/2)}{\sin(\theta/2)} e^{i\varphi} a_1$$

normalized eigenvector:

$$|s_2\rangle = \begin{pmatrix} -\sin(\theta/2) \\ \cos(\theta/2)\, e^{i\varphi} \end{pmatrix}$$

$$|s\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad |s\rangle = c_1 |s_1\rangle + c_2 |s_2\rangle$$

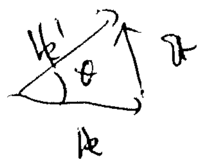
$$c_1 = \langle s_1 | s \rangle = \left(\cos(\theta/2),\ \sin(\theta/2)\, e^{-i\varphi}\right) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \boxed{\cos\left(\frac{\theta}{2}\right)}$$

$$P_1 = |c_1|^2 = \cos^2(\theta/2)$$

Scattering amplitude $f = -\frac{m}{2\pi\hbar^2}\int d^3r\, U(\mathbf{r})\, e^{-i\mathbf{q}\cdot\mathbf{r}}$

where $\mathbf{q} = \mathbf{k} - \mathbf{k}'$ $\quad |\mathbf{k}| = |\mathbf{k}'| = k$

$q = 2k\sin\frac{\theta}{2}$.



For Yukawa potential. $U = \frac{\alpha}{r} e^{-r/\alpha}$.

$$f = -\frac{m}{2\pi\hbar^2}\int_0^{+\infty} dr' \overbrace{\int_0^{2\pi} d\phi}^{2\pi} \int_0^{\pi} \underbrace{d\theta' \sin\theta'}_{d\cos\theta'}\, r' \cdot \frac{\alpha}{r'} e^{-r'/\alpha} \cdot e^{-iqr\cos}$$

$$= +\frac{m\alpha}{2\pi\hbar^2}\cdot 2\pi \left[\int_0^{+\infty} dr' \frac{e^{-iqr'-r'/\alpha}}{+iqr} - \int_0^{+\infty} dr' \frac{e^{iqr'-r'/\alpha}}{iqr}\right]$$

$$= \frac{m\alpha}{\hbar^2}\,\frac{2iq}{\left(\frac{1}{\alpha}\right)^2 + q^2}$$

[where $q^2 = (\mathbf{k}-\mathbf{k}')^2 = k^2 - \underbrace{2\mathbf{k}\cdot\mathbf{k}'}_{= 2k^2\cos\theta} + k^2$

$= 2k^2(1-\cos\theta) = 4k^2\sin^2\frac{\theta}{2}$.

$$= \frac{m\alpha^3}{\hbar^2}\cdot\frac{2iq}{1 + 4k^2\sin^2\frac{\theta}{2}}$$ //

$$\frac{d\sigma}{d\Omega} = |f(\theta)|^2$$

$$= \frac{4m^2 g^2 \alpha^6}{\hbar^4} \cdot \frac{1}{1 + 4g^2\alpha^2 \underbrace{\sin^2\frac{\theta}{2}}_{(1-\cos\theta)}}$$

Total cross section

$$\sigma = \int_0 |f(\theta)|^2 \, 2\pi \underbrace{\sin\theta d\theta}_{d\cos\theta}$$

$$= \frac{4m^2 g^2 \alpha^6}{\hbar^2} 2\pi \int_0^\pi \frac{d(\cos\theta')}{1 + 4k^2\alpha^2(1-\cos\theta')}$$

$$= \frac{16\pi \alpha b^2 m^2}{\hbar^4} \frac{1}{4k^2\alpha^2 + 1}$$

the fact that the nearer E is to V in (4.30) the smaller ϕ is. This last fact can be seen analytically from

$$\tan\phi = \frac{\kappa_2}{k_1} = \left(\frac{V - E}{E}\right)^{1/2} \tag{4.31}$$

4.2 POTENTIAL WELLS

We shall now use the same methods to study a situation in which stationary energy states occur and which explains qualitatively the reason for atomic and nuclear energy levels. Let the potential be in the form of a rectangular well, as in Fig. 4.5. At the same time, by making V positive, we can study the penetration of a potential barrier of finite width, as opposed to the infinite barrier of Fig. 4.1. We shall therefore investigate three cases: The first is when there is a beam of particles of energy E ($E > 0$) incident from the left and V is either positive or negative but $V < E$. The second is when V is high enough ($V > E$) for the central region, now become a barrier, to be forbidden from the classical point of view; and the third is when V is again negative and E is also negative, so that the regions to right and left are forbidden and the particle is confined in the neighborhood of the well.

CASE 1. $V < E$, $E > 0$. The wave functions in the three regions are, as before,

Region I: $\quad \psi_1 = Ae^{ik_1x} + Be^{-ik_1x} \qquad \hbar k_1 = (2mE)^{1/2}$ (4.32)

Region II: $\quad \psi_2 = Ce^{ik_2x} + De^{-ik_2x} \qquad \hbar k_2 = [2m(E - V)]^{1/2}$ (4.33)

Region III: $\quad \psi_3 = Fe^{ik_1x}$ (4.34)

and the boundary conditions at $x = -a$ and $x = +a$ give the four equations

$$Ae^{-ik_1a} + Be^{ik_1a} = Ce^{-ik_2a} + De^{ik_2a}$$

$$k_1(Ae^{-ik_1a} - Be^{ik_1a}) = k_2(Ce^{-ik_2a} - De^{ik_2a})$$

$$Ce^{ik_2a} + De^{-ik_2a} = Fe^{ik_1a}$$

$$k_2(Ce^{ik_2a} - De^{-ik_2a}) = k_1Fe^{ik_1a}$$

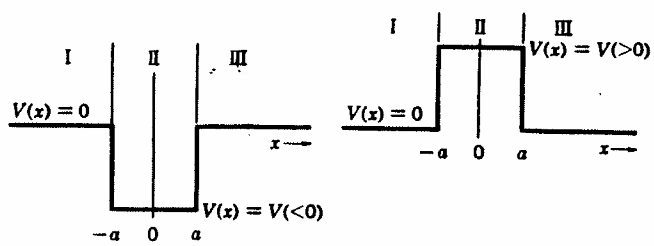


FIGURE 4.5
Rectangular potential well and barrier, showing the three regions of integration.

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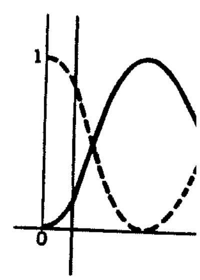


FIGURE 4.6
Transmission and r

These equations are solved by the method of determinants (or by solving the last two for C and D in terms of F, putting these into the first two and finding A and B in terms of F). The results are

$$A = Fe^{2ik_1 a}\left[\cos 2k_2 a - \frac{1}{2}i\left(\frac{k_1}{k_2} + \frac{k_2}{k_1}\right)\sin 2k_2 a\right] \tag{4.35}$$

$$B = \frac{1}{2}iF\left(\frac{k_2}{k_1} - \frac{k_1}{k_2}\right)\sin 2k_2 a \tag{4.36}$$

The first of these gives the transmission coefficient of the entire barrier, most conveniently expressed in terms of its reciprocal:

$$\begin{aligned}\frac{1}{T} = \left|\frac{A}{F}\right|^2 &= \cos^2 2k_2 a + \frac{1}{4}\left(\frac{k_1}{k_2} + \frac{k_2}{k_1}\right)^2 \sin^2 2k_2 a \\ &= 1 + \frac{1}{4}\left(\frac{k_1}{k_2} - \frac{k_2}{k_1}\right)^2 \sin^2 2k_2 a\end{aligned} \tag{4.37}$$

or, finally, using the definitions of k_1 and k_2,

$$\frac{1}{T} = 1 + \frac{1}{4}\frac{V^2}{E(E-V)}\sin^2 2k_2 a \tag{4.38}$$

The reflection coefficient is

$$R = 1 - T \tag{4.39}$$

Problem 4.1. Let $V < 0$ (potential well). Evaluate T in the limit $E \to 0$. Make sure your answer is valid for all (negative) values of V. Show how one should be able to see the answer without doing the whole calculation.

Problem 4.2. Find R directly and verify that it satisfies (4.39). Explain by a physical argument why there are positive energies for which $R = 0$ but none for which $T = 0$.

The qualitative behavior of T and R as functions of the energy E is sketched in Fig. 4.6.

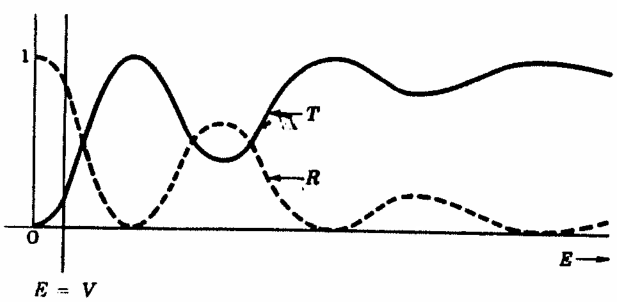


FIGURE 4.6
Transmission and reflection coefficients of a rectangular potential barrier.

The most immediate application of this theory is to the collision of slow electrons with atoms. Although the situation is here three-dimensional, at low energies it is reasonable to compare it to the encounter of an electron with a potential well in one dimension. The attractive potential exists because as the impinging electron enters the atom, the nuclear charge is no longer screened from it by the atomic electrons. Since R is a measure of the obstruction offered by the obstacle, one would expect that the scattering cross section for slow electrons would go through a minimum at some point. The results of measurements on rare-gas atoms in the low-energy range are shown in Fig. 4.7. The striking decrease in cross section, as a result of which the gases are almost transparent to electrons of about 1 eV, was discovered by Ramsauer in 1920 and independently by Townsend and Bailey later. It seems to be characteristic of many elements, subject only to the difficulty of obtaining monatomic vapors. The absence of further maxima and minima is due to complications involving states of differing angular momenta which set in at higher energies.

CASE 2. $V > E > 0$. The treatment in case 1 must be changed when the barrier becomes high enough to be impenetrable from the classical point of view. This is mirrored in the fact that k_2 in (4.33) now becomes imaginary. As in the last section, we write it as $i\kappa_2$ and note that none of the algebraic details of the theory are changed. We can therefore rewrite (4.35) and (4.36) at once (using $\sin ix = i \sinh x$, $\cos ix = \cosh x$) as

$$A = Fe^{2ik_1 a}\left[\cosh 2\kappa_2 a - \frac{1}{2}i\left(\frac{k_1}{\kappa_2} - \frac{\kappa_2}{k_1}\right)\sinh 2\kappa_2 a\right] \tag{4.40}$$

$$B = -\frac{1}{2}iF\left(\frac{\kappa_2}{k_1} + \frac{k_1}{\kappa_2}\right)\sinh 2\kappa_2 a \tag{4.41}$$

from which

$$\frac{1}{T} = \left|\frac{A}{F}\right|^2 = \cosh^2 2\kappa_2 a + \frac{1}{4}\left(\frac{k_1}{\kappa_2} - \frac{\kappa_2}{k_1}\right)^2 \sinh^2 2\kappa_2 a$$

$$= 1 + \frac{1}{4}\left(\frac{k_1}{\kappa_2} + \frac{\kappa_2}{k_1}\right)^2 \sinh^2 2\kappa_2 a$$

or

$$\frac{1}{T} = 1 + \frac{1}{4}\frac{V^2}{E(V-E)}\sinh^2 2\kappa_2 a \tag{4.42}$$

This is the formula for the transparency of a barrier which, classically speaking, should not be transparent at all. [The possibility of such transparency was mentioned in the remarks following (4.31).] It gives rise to a number of interesting and typically quantum-mechanical effects such as the slow emission of

CASE 3. $V < E < 0$. Here the wave function must decay exponentially on both sides of the potential well, and the problem of matching solutions is somewhat different. It is illustrated in Fig. 4.9*a*, where for all given values of a, V, and E we can attain a smooth matching at the left side of the well but have no further adjustable parameters available to make ψ and $d\psi/dx$ continuous at the right. Only for certain values of E will a smooth match be possible at both edges of the well, and it is therefore only for these values that allowable solutions of (4.1) exist. Three possibilities, corresponding to successively higher values of E, are shown in Fig. 4.9*b*. In the case illustrated, only these three values of E are possible. In actual cases, the number of energy states may be finite (for example, one for the deuteron, cf. Chap. 19) or infinite, as for hydrogen (Chap. 6). The calculation of E cannot be carried out in closed form; since the graphical method ordinarily used is of no general interest, we omit it here. We shall encounter a more interesting example of the same procedure in Sec. 19.1. I mention here only a simple limiting case, already discussed at the end of Chap. 1, in which the well becomes infinitely deep ($V \rightarrow -\infty$). The wave function must vanish (why?) at each edge, and the only ones that satisfy these boundary conditions correspond to energies

$$E = \frac{(n\pi\hbar)^2}{2ma^2} \qquad n = 1, 2, 3, \ldots \tag{4.43}$$

A few of the wave functions are shown in Fig. 4.10.

Problem 4.4. Why are the nonpositive values of n not included in (4.43)?

One has in case 3 the basic reason given by quantum mechanics for the occurrence of definite energy states in nature. They are a standing-wave phenomenon, as was first suggested by de Broglie. The relation of energy values to the

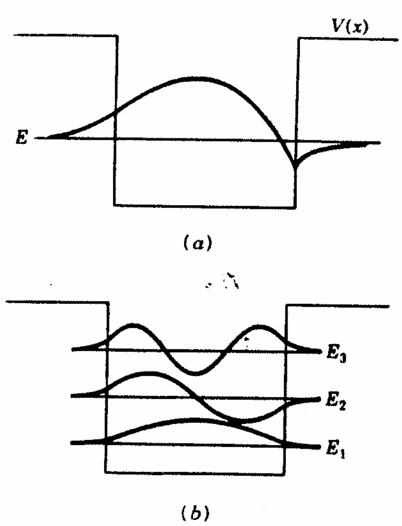
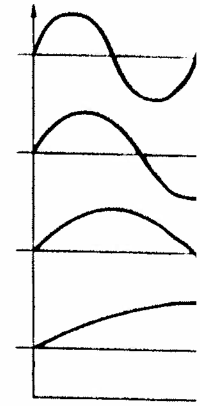


FIGURE 4.9
(*a*) Wave function of a particle in a potential well, showing the difficulty of matching ψ and its derivative at both sides of the well when E is chosen arbitrarily. (*b*) Properly matched eigenfunctions, drawn about the corresponding energy levels.



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II-6

$$V(t) = \lambda x\, e^{-\alpha t^2} = \lambda e^{-\alpha t^2} \sqrt{\frac{\hbar}{2m\omega}}\,(a + a^\dagger)$$

$$c_1^{(1)}(\infty) = \lambda \frac{-i}{\hbar} \sqrt{\frac{\hbar}{2m\omega}} \int_{-\infty}^{\infty} dt'\, e^{-\alpha t'^2 + i\omega t'} \langle 1 | a + a^\dagger | 0 \rangle$$

$$= \lambda \frac{-i}{\hbar} \sqrt{\frac{\hbar}{2m\omega}} \sqrt{\frac{\pi}{\alpha}}\, e^{-\omega^2/4\alpha}$$

$$P(0 \to 1) = \frac{\pi}{\alpha} \lambda^2 \frac{1}{2m\hbar\omega} e^{-\omega^2/2\alpha}$$

II-7 [10]

Two blocks are made of identical metal of specific heat (per unit mass) c and are isolated from the rest of the outside world, but not from each other. Initially, the first block, with mass $2m$, has temperature T; the second block, with mass m, has temperature $4T$. They are then brought into contact with each other, and eventually they come to thermal equilibrium with each other.

What is the total change in entropy ΔS_{total} for the system (the two blocks combined)? You must express your answer in terms of m, c, and T (not all of them may show up in your final answer).

$$Q_1 + Q_2 = 0 \quad \Rightarrow \text{they will reach common/final temperature } T_f$$

$$2mc\,(T_f - T) + mc\,(T_f - 4T) = 0$$

$$3mcT_f = 6mcT \quad \Rightarrow \quad \boxed{T_f = 2T}$$

For each body:

$$\left.\begin{array}{l} \delta Q = T\,dS \\ \delta Q = \underset{\text{(respective mass)}}{m}\,c\,dT \end{array}\right\} \Rightarrow dS = mc\frac{dT}{T}$$

$$\text{block ①: } \Delta S_1 = (2m)\,c \cdot \int_T^{2T} \frac{dT}{T} = 2mc \ln\left(\frac{2T}{T}\right) = 2mc\ln(2) \quad (>0)$$

$$\text{block ②: } \Delta S_2 = mc \cdot \int_{4T}^{2T} \frac{dT}{T} = mc \ln\left(\frac{2T}{4T}\right) = mc\ln(1/2) \quad (<0)$$

$$\Delta S_{tot} = \Delta S_1 + \Delta S_2 = 2mc\ln(2) + mc\ln(1/2) =$$

$$= mc\{2\ln(2) - \ln(2)\} = \boxed{mc\ln(2)} \quad (>0)$$

II-8 [10]

Consider the Van der Waals gas given by the equation of state and energy given by

$$\left(P+\frac{a}{v^2}\right)(v-b)=RT \quad ,$$

$$u(T,v)=c_v T-\frac{a}{v} \quad ,$$

where a, b, R, and c_v are positive constants. The gas undergoes a reversible (quasi-static) adiabatic expansion from v to $2v$. Obtain the ratio of the final and initial temperatures, T_2/T_1. You must express your answer in terms of the variables and constants given above (i.e., do not reinvent new constant or redefine new variables)

$$du = C_v\,dT + \frac{a}{v^2}\,dv \qquad P = \frac{RT}{v-b} - \frac{a}{v^2}$$

$$du = \delta Q - P\,dv \qquad \text{and } \underline{\delta Q = 0}$$

$$du + P\,dv = 0$$

$$C_v\,dT + \frac{a}{v^2}\,dv + \left\{\frac{RT}{v-b} - \frac{a}{v^2}\right)dv = 0$$

$$C_v\,dT + \frac{RT}{v-b}\,dv = 0$$

$$C_v\frac{dT}{T} + R\frac{dv}{v-b} = 0 \implies C_v \ln(T) + R\ln(v-b) = \text{const.}$$

$$\implies \boxed{T\cdot(v-b)^{R/C_v} = \text{const.}} \quad \text{(}T-v \text{ adiabatic curve for the V.d.W gas}$$

$$\frac{T_2}{T_1} = \left(\frac{v_1-b}{v_2-b}\right)^{R/C_v} = \boxed{\left(\frac{v-b}{2v-b}\right)^{R/C_v}}$$

$\underline{\text{II}-9}$

$$Z = \int dp\, dq\, \exp\left(-\frac{\beta p^2}{2m} - \beta \frac{m\omega^2}{2} q^2\right)$$

$$= \frac{2\pi}{\beta\omega} = \frac{2\pi kT}{\omega}$$

$$F = -kT \ln \frac{2\pi kT}{\omega}$$

$$S = -\frac{dF}{dT} = k\left(\ln \frac{2\pi kT}{\omega} + 1\right)$$

II - 10

Problem II-10 Neutron gas

Solve for the pressure of a neutron gas at $T = 0$. How does it compares to the pressure of an ideal gas under similar N, T, V conditions?

Solution Protons are $s = 1/2$ Fermions. At $T = 0$ the the chemical potential is equal to the Fermi energy. The Fermi energy is defined by the number of particles in the system

$$N = \frac{(2s+1)V}{h^3} \int d^3p = \frac{(2s+1)V}{h^3}\frac{4\pi}{3}p_F^3$$

where p_F is the Fermi momentum. The Fermi energy is

$$E_F = \frac{p_F^2}{2m} = \frac{h^2}{2m}\Big(\frac{3}{4\pi(2s+1)}\Big)^{2/3}$$

The energy of the system is

$$E = \frac{(2s+1)V}{h^3}\int d^3p(\frac{p^2}{2m}) = \frac{3}{5}NE_F \sim V^{-2/3}$$

The pressure is

$$p = -\Big(\frac{\partial E}{\partial V}\Big)_{T,N} = \frac{2}{3}\frac{E}{V} = \frac{2}{5}\frac{N}{V}E_F \neq 0$$

For an ideal gas, $PV = N\kappa_B T = 0$, at $T = 0$.