Physics PhD Qualifying Examination
Part I – Wednesday, January 20, 2010

Name:____

(please print)

Identification Number:________

STUDENT: Designate the problem numbers that you are handing in for grading in the appropriate left hand boxes below. Initial the right hand box.

PROCTOR: Check off the right hand boxes corresponding to the problems received from each student. Initial in the right hand box.

	1	
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	6	
	7	
	8	
	9	
	10	

Student's initials
problems handed in:
Proctor's initials

INSTRUCTIONS FOR SUBMITTING ANSWER SHEETS

1. DO NOT PUT YOUR NAME ON ANY ANSWER SHEET. EXAMS WILL BE COLLATED AND GRADED BY THE ID NUMBER ABOVE.
2. Use at least one separate preprinted answer sheet for each problem. Write on only one side of each answer sheet.
3. Write your **identification number** listed above, in the appropriate box on each preprinted answer sheet.
4. Write the **problem number** in the appropriate box of each preprinted answer sheet. If you use more than one page for an answer, then number the answer sheets with both problem number and page (e.g. Problem 9 – Page 1 of 3).
5. Staple together all the pages pertaining to a given problem. Use a paper clip to group together all eight problems that you are handing in.
6. Hand in a total of *eight* problems. A passing distribution will normally include at least three passed problems from problems 1-5 (Mechanics) and three problems from problems 6-10 (Electricity and Magnetism). **DO NOT HAND IN MORE THAN EIGHT PROBLEMS.**
7. **YOU MUST SHOW ALL YOUR WORK.**

[I-1] **[2,8]**

A nonuniform wire of total length L and total mass M has a *variable linear mass density* given by $\mu(x) = kx$, where x is the distance measured from one end of the wire and k is a constant. The tension in the wire is F.

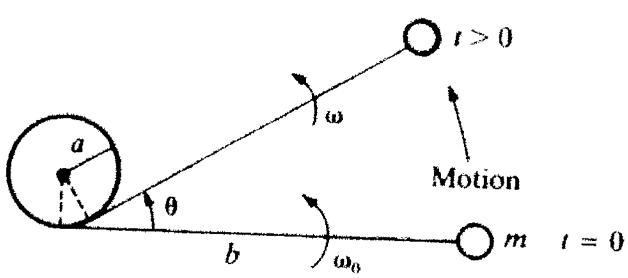
(a) Determine k in terms of M and L.

(b) How long does it take for a transverse pulse generated at one end of the wire to travel to the other end? You must express your answer in terms of M, L, and F.

[I-2] **[4,2,4]**

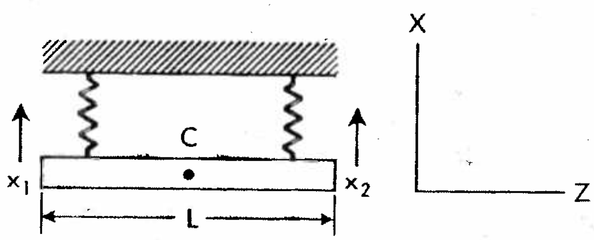
A particle of mass m at the end of a light string wraps itself about a fixed vertical cylinder of radius a (see figure below). All the motion is in the horizontal plane (so disregard gravity). The angular velocity of the cord is ω_o when the distance from the particle to the point of contact of the string is b.

(a) Write the Lagrangian for the motion of the particle in terms of θ (angle defined in figure below) and $\dot{\theta}$.
(b) Using the Lagrangian, determine the equation of motion in its simplest form.
(c) Find the angular velocity after the cord has turned through an angle θ.



[I-3] **[10]**

A rigid uniform bar of mass M and length L is supported in equilibrium in a horizontal position by two mass-less springs attached one at each end as shown in the figure below.



The springs have the same force constant k. The motion of the center of gravity is constrained to move parallel to the vertical X axis. Find the *normal modes* and *frequencies of vibration* of the system, if the motion is constrained to the XZ-plane.

[I-4] **[10]**

A system of particles consists of two masses m_1 and m_2. The two masses move in a plane and interact by a potential energy $U(r) = kr^2$. The position of mass m_1 is $\boldsymbol{r}_1$ and the position of mass m_2 is $\boldsymbol{r}_2$. The relative distance between m_1 and m_2 is $\boldsymbol{r} = \boldsymbol{r}_2 - \boldsymbol{r}_1$ and r is the magnitude of $\boldsymbol{r}$.

(a) Find the center of mass $\boldsymbol{R}$ of the system

(b) Write down the Lagrangian of the system in terms of the center of mass $\boldsymbol{R}$ and relative position $\boldsymbol{r}$.

(c) Find Lagrange's equations of motion for the center-of-mass coordinates X, Y and relative coordinates x, y.

(d) Solve for Lagrange's equations of motion for the center-of-mass coordinates X(t), Y(t) and the relative coordinates $x(t)$, $y(t)$.

[I-5] **[10]**

Muons are unstable particles created upon entering the Earth atmosphere at a speed $v = 0.98c$ ($c = 3\times 10^{8}$ m/s). If observed in an inertial frame where the muons are at rest (e.g., in a lab), their number decays according to the radioactive decay law

$$N(t) = N(0)\,2^{-t/t_{1/2}} = N(0)\,e^{-\ln(2)t/t_{1/2}},$$

where $t_{1/2} = 1.5\mu s$ is the "half life" (or lifetime).

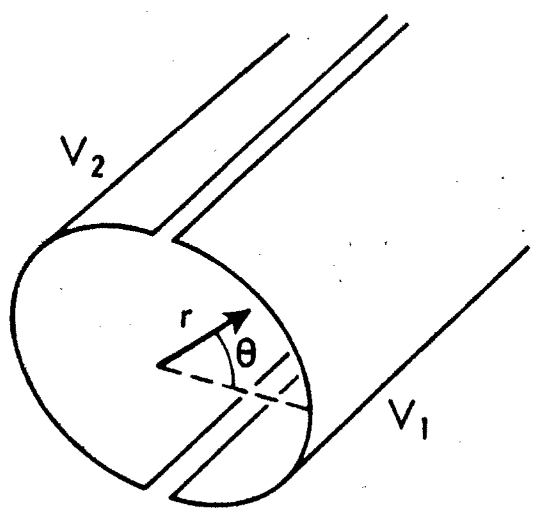
The number of muons detected by a detector at a top of a mountain at height $h = 1000\,\text{m}$ is $N(0) = 1{,}000$. Of these muons, how many will reach sea level (at 0m)? (The height of the mountain is measured by a stationary observer on the Earth).

[I-6] **[10]**

A long hollow cylindrical conductor of radius a is divided into two parts by a plane through the axis, and the parts are separated by a small interval. If the two parts are kept at potentials V_1 and V_2, show that the potential at any point within the cylinder is given by

$$V(r,\theta) = \frac{V_1 + V_2}{2} + \frac{2(V_1 - V_2)}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(2n-1)}\left(\frac{r}{a}\right)^{2n-1}\cos[(2n-1)\theta],$$

where r is the distance from the axis of the cylinder, and θ is the corresponding angle in cylindrical coordinates, as shown in the figure below.



[I-7] **[10]**

(a) Write down Maxwell's equations assuming that no dielectrics or magnetic materials are present. State your system of units.

In all of the following you must justify your answer.

(b) If the signs of all the source charges are reversed, what happens to the electric and magnetic field $\boldsymbol{E}$ and $\boldsymbol{B}$?

(c) If the system is space inverted, i.e., $x \to x' = -x$, $y \to y' = -y$, and $z \to z' = -z$, what happens to the charge density ρ, the current density $\boldsymbol{j}$, and the electric and magnetic fields $\boldsymbol{E}$ and $\boldsymbol{B}$?

(d) If the system is time reversed, i.e., $t \to t' = -t$, what happens to ρ, $\boldsymbol{j}$, $\boldsymbol{E}$ and $\boldsymbol{B}$?

[I-8] **[10]**

Two loops, each of radius R and carrying a current i are placed parallel to the YZ plane and centered about the x-axis at $x = -l$ and $x = +l$, respectively. Each produces a magnetic field in the $+x$ direction along its axis.

(a) Find an expression for the magnetic field along the x-axis.

(b) Show that the first derivative of B_x with respect to x is zero at $x = 0$.

(c) For which ratio of l to R is the second derivative of B_x with respect to x also equal to zero at $x = 0$?

[I-9] **[10]**

The scalar potential V and vector potential $\boldsymbol{A}$ of a perfect electrical dipole $\boldsymbol{p}(t) = p_o \cos(\omega t)\hat{z}$ in the far field $(1/r) << \omega/c$ are described by equations (1) and (2):

$$V(r,\theta,t) = -\frac{p_o\omega}{4\pi\,\varepsilon_o c}\frac{\cos(\theta)}{r}\sin[\omega(t - r/c)] \ , \qquad (1)$$

$$\boldsymbol{A}(r,\theta,t) = -\frac{\mu_o p_o \omega}{4\pi}\frac{1}{r}\sin[\omega(t - r/c)]\hat{z} \ . \qquad (2)$$

(a) Calculate the electric field $\boldsymbol{E}$ and the magnetic induction $\boldsymbol{B}$ of the electrical dipole in the far field.
(b) Calculate the total power P radiated by the electrical dipole in the far field.

Hint: $\hat{z} = \cos(\theta)\hat{\boldsymbol{r}} - \sin(\theta)\hat{\boldsymbol{\theta}}$.

[I-10] **[10]**

Using the fact that $k^\alpha = (\frac{\omega}{c}, \boldsymbol{k})$ is a four-vector, apply the Lorentz transformation to this quantity to derive the Doppler shift for a light wave, that is, the frequency observed in the laboratory for a source moving in the $+x$ direction with velocity v=βc for arbitrary $\boldsymbol{k}$.

Solutions Part I

I-1 $\mu(x) = kx$ M, L

a) $$M = \int_0^L \mu(x)\,dx = \int_0^L kx\,dx = k\frac{L^2}{2}$$

$$k = \frac{2M}{L^2}$$

b) $$v = \sqrt{\frac{F}{\mu}}$$ $F = \text{const.}$, $\mu(x)$

$$\Rightarrow \quad v(x) = \sqrt{\frac{F}{\mu(x)}} \qquad v(x) = \frac{dx}{dt}$$

$$\frac{dx}{dt} = \sqrt{\frac{F}{\mu(x)}}$$

$$dt = \sqrt{\frac{\mu(x)}{F}}\,dx$$

$$T = \int_0^T dt = \int_0^L \sqrt{\frac{\mu(x)}{F}}\,dx = \sqrt{\frac{2M}{FL^2}}\int_0^L x^{1/2}\,dx = \sqrt{\frac{2M}{FL^2}}\,\frac{2}{3}L^{3/2}$$

$$\boxed{= \sqrt{\frac{8ML}{9F}}}$$

Solution I.2 (A) Basic geometry gives the coordinates with respect to the bottom of the cylinder:

$$x = a\sin\theta + (b - a\theta)\cos\theta, \quad y = a(1 - \cos\theta) + (b - a\theta)\sin\theta$$

From this it is easy to compute

$$L = T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) = \frac{1}{2}m(b - a\theta)^2\dot{\theta}^2$$

(B) Using this Lagrangian, the equation of motion is

$$m(b - a\theta)^2\ddot{\theta} - ma(b - a\theta)\dot{\theta}^2 = 0$$

This simplifies to

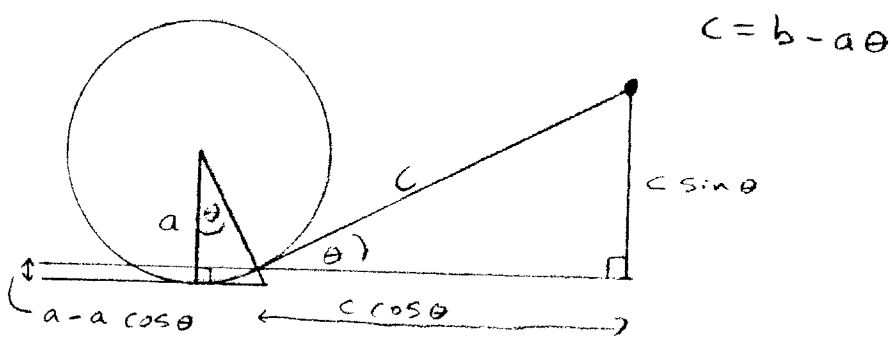
$$(b - a\theta)\ddot{\theta} - a\dot{\theta}^2 = 0$$

(C) Now note that the equation of motion can be written as

$$\frac{d}{dt}[(b - a\theta)\dot{\theta}] = 0$$

Thus $(b - a\theta)\dot{\theta}$ = constant. The constant is determined from the initial condition, yielding

$$\omega(\theta) = \frac{b\omega_0}{b - a\theta}$$



Q.E. Ph.D. Jan. 2010

(I-3) Solutions:

Let the vertical displacements from the equilibrium positions be X_1, X_2. For the motion of the center of mass C, $\vec{F} = m\vec{a}$ yields

$$\frac{M}{2}(\ddot{X}_1 + \ddot{X}_2) = -k(X_1 + X_2) - mg \ ,$$

and the torque conditions gives (for small X_1, X_2)

$$\frac{I_0}{L}(\ddot{X}_2 - \ddot{X}_1) = -\frac{L}{2}k(X_2 - X_1)$$

and I_0 = moment of inertia about C $= ML^2/12$. The gravity term, which merely determines the unextended spring lengths, can be transformed away. It does not affect the modes. From the above equations there are obviously two modes:

(a) A symmetric one, $X_1 = x_1 + x_2$, with frequency $\omega_s^2 = 2k/M$, and

(b) an antisymmetric one, $X_2 = x_1 - x_2$, and frequency, $\omega_a^2 = 6k/M$.

$\boxed{\text{I-4}}$

- $\vec{R} = \dfrac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{m_1 + m_2}$ center of mass

- $L = T - U = \frac{1}{2} m_1 \dot{\vec{r}}_1^{\,2} + \frac{1}{2} m_2 \dot{\vec{r}}_2^{\,2} - k r^2$

coordinate transformation

$$\begin{matrix} \vec{R}(\vec{r}_1, \vec{r}_2) \\ \vec{r} = \vec{r}_2 - \vec{r}_1 \end{matrix} \Rightarrow \begin{matrix} \vec{r}_1(\vec{R}, \vec{r}) \\ \vec{r}_2(\vec{R}, \vec{r}) \end{matrix}$$

$\rightsquigarrow L = \frac{1}{2} M \dot{\vec{R}}^2 + \frac{1}{2} \mu \dot{\vec{r}}^2 - k r^2$ with $M = m_1 + m_2$

$\mu = \dfrac{m_1 m_2}{m_1 + m_2}$

center of mass motion

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\vec{R}}} - \frac{\partial L}{\partial \vec{R}} = 0 \Rightarrow M\ddot{\vec{R}} = 0 \Rightarrow \begin{matrix} M\ddot{X} = 0 \\ M\ddot{Y} = 0 \end{matrix}$$

$X(t) = Et + F$

$Y(t) = Gt + H$

relative motion

$x = r\cos\theta$ $\quad \dot{\vec{r}}^2 = \dot{r}^2 + r^2\dot{\theta}^2$

$y = r\sin\theta$

$$L_{rel} = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\theta}^2) - k r^2$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{r}} - \frac{\partial L}{\partial r} = \mu\ddot{r} - \mu r\dot{\theta}^2 + 2kr = 0$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = 0 \qquad \mu r^2\ddot{\theta} = 0$$

or in x and y:

$$L_{rel} = \frac{1}{2}\mu\dot{r}^2 - kr^2 = \frac{1}{2}\mu(\dot{x}^2 + \dot{y}^2) - k(x^2 + y^2)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = \mu\ddot{x} + 2kx = 0 \Rightarrow \ddot{x} = -\frac{2k}{\mu}x$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{y}} - \frac{\partial L}{\partial y} = \mu\ddot{y} + 2yk = 0 \Rightarrow \ddot{y} = -\frac{2k}{\mu}$$

with $\omega = \sqrt{\dfrac{2k}{\mu}}$ $\quad x(t) = A\cos\omega t + B\sin\omega t$

$y(t) = C\cos\omega t + D\sin\omega t$

$\boxed{I-5}$

$$t = \frac{h}{v} = \frac{h}{0.98c} = \frac{1000\,m}{0.98 \times 3 \times 10^{8}\frac{m}{s}} = 3.4 \times 10^{-6}\,s = 3.4\,\mu s$$

t: time of travel between top of the mountain and sea level for a muon, as observed from Earth.

t_0: time of travel between the same two points observed in muon's rest frame.

$$t = \gamma t_0 = \frac{1}{\sqrt{1 - v^2/c^2}}\, t_0 \qquad \text{(time dilation)}$$

$$t_0 = \sqrt{1 - v^2/c^2}\; t = 6.77 \times 10^{-7}\,s = 0.677\,\mu s \quad \text{(in muon frame)}$$

$$N(t_0) = N_0\, e^{-\ln(2)\, t_0/t_{1/2}} = 1{,}000 \times 0.731 \simeq \boxed{731}$$

# of muons at sea level

Q.E. Ph.D. Jan. 2010

(I-6). Solution:

Begin with the equation

$$\nabla^2 V = 0 = \frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} + \frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2}.$$

We separate variables and find

$$V = \sum_{m=0}^{\infty} \left(\frac{r}{a}\right)^m [A_m \cos(m\theta) + B_m \sin(m\theta)].$$

Now

$$V(r = a, \theta) = \sum_{m=0}^{\infty} [A_m \cos(m\theta) + B_m \sin(m\theta)].$$

$B_m = 0$ by symmetry since $V(\theta) = V(-\theta)$. A_m may be calculated when $V(a, \theta)$ is multiplied by $\cos(n\,\theta)$, and integrated over θ from zero to 2π; then

$$\int_{-\pi/2}^{\pi/2} V_1 \cos n\theta d\theta + \int_{\pi/2}^{3\pi/2} V_2 \cos n\theta d\theta = \frac{2(V_1 - V_2)}{n} \sin\left(\frac{n\pi}{2}\right) = \pi A_n,$$

and $\pi(V_2 + V_1) = 2\pi A_0$. Thus

$$V = \frac{(V_1 + V_2)}{2} + \frac{2(V_1 - V_2)}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)}\left(\frac{r}{a}\right)^{2n-1} \cos[(2n-1)\theta].$$

Finally, it is worthwhile to note that the series expansion may be summed. This is accomplished by writing

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{2n-1}}{(2n-1)} \cos[(2n-1)\theta] = \operatorname{Re} \int_0^x dy \left(-\frac{i}{y}\right) \sum_{n=1}^{\infty} (iye^{i\theta})^{2n-1},$$

which, after making use of the expansion

$$\frac{1}{(1-x)} = \sum_{n=0}^{\infty} x^n \qquad \text{for} \qquad |x| < 1,$$

becomes

$$\operatorname{Re} \int_0^x dy \left(\frac{-i}{2y}\right) \left\{\frac{1}{1 - iye^{i\theta}} - \frac{1}{1 + iye^{i\theta}}\right\}.$$

Upon completing the integration, we obtain

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)} x^{2n-1} \cos[(2n-1)\theta] = \frac{1}{2} \operatorname{Im} \log \left\{\frac{1 + ixe^{i\theta}}{1 - ixe^{i\theta}}\right\}$$

$$= \frac{1}{2} \tan^{-1} \left\{\frac{2x \cos\theta}{1 - x^2}\right\}.$$

Thus

$$V(r, \theta) = \frac{(V_1 + V_2)}{2} + \frac{(V_1 - V_2)}{\pi} \tan^{-1} \left\{\frac{2ar \cos\theta}{a^2 - r^2}\right\}.$$

This closed-form expression can also be obtained through the use of Green's-function techniques.

(I-7)

0/2

(a) In MKS system

Maxwell's equations are

(i) $\nabla\cdot\vec{E}=\frac{\rho}{\varepsilon_0}$ (iii) $\nabla\times\vec{E}=-\frac{\partial\vec{B}}{\partial t}$

(ii) $\nabla\cdot\vec{B}=0$ (iv) $\nabla\times\vec{B}=\mu_0\vec{j}+\frac{1}{\mu_0\varepsilon_0}\frac{\partial\vec{E}}{\partial t}$ #

(b) under charge reversed
$$\begin{cases} e\longrightarrow e'=-e \\ \rho\longrightarrow \rho'=-\rho \\ \vec{j}\longrightarrow \vec{j}'=-\vec{j} \\ \nabla\longrightarrow \nabla'=\nabla \\ \frac{\partial}{\partial t}\longrightarrow \frac{\partial}{\partial t'}=\frac{\partial}{\partial t} \end{cases}$$

Maxwell's equations become

(i) $\nabla'\cdot\vec{E}'=\frac{\rho'}{\varepsilon_0}$ (iii) $\nabla'\times\vec{E}'=-\frac{\partial\vec{B}'}{\partial t'}$

(ii) $\nabla'\cdot\vec{B}'=0$ (iv) $\nabla\times\vec{B}'=\mu_0\vec{j}'+\frac{1}{\mu_0\varepsilon_0}\frac{\partial\vec{E}'}{\partial t'}$

$\Rightarrow$ comparison of (a)-(i) and (b)-(i)

$$\underline{\underline{\vec{E}'=-\vec{E}}} \quad \text{——} (1)$$

substituting (1) into (b)-(iv), we get

$\Rightarrow \nabla'\times\vec{B}'=-\mu_0\vec{j}-\frac{1}{\mu_0\varepsilon_0}\frac{\partial\vec{E}}{\partial t} \quad\Rightarrow\quad \underline{\underline{\vec{B}'=-\vec{B}}}$ #

(c) under space inversion

$\vec{r} \longrightarrow \vec{r}' = -\vec{r}$ $\qquad \nabla \longrightarrow \nabla' = -\nabla$

$\frac{\partial}{\partial t} \longrightarrow \frac{\partial}{\partial t} = \frac{\partial}{\partial t}$

$e \rightarrow e' = e \quad \Rightarrow \underline{\rho \rightarrow \rho' = \rho}$

$\Rightarrow \vec{j} \rightarrow \vec{j}' = \rho' \vec{u}'$

$= \rho(-\vec{u})$

$\underline{= -\vec{j}}$

where $\vec{u}$ is the velocity of the charge

check (a)-(i) and (a)-(iii), we can get

$\vec{E}' = -\vec{E}$ and $\vec{B}' = \vec{B}$ #

(d) under time reverse

$\frac{\partial}{\partial t} \longrightarrow \frac{\partial}{\partial t'} = -\frac{\partial}{\partial t}$, $\nabla \rightarrow \nabla' = \nabla$, $e \rightarrow e' = e$

$\Rightarrow \rho' = \rho$ and $\vec{j}' = -\vec{j}$

check (a)-(i) and (a)-(ii), we can get

$\vec{E}' = \vec{E}$ and $\vec{B}' = -\vec{B}$

#

$\boxed{I-8}$ Biot-Savart problem: AG

$$B_x = \oint \frac{l\,dl}{R^2+(l+)^2}\sin\theta_1 + \oint \frac{l'dl'}{R^2+(l-x)^2}\sin\theta_2$$

(i) $$B_x = 2\pi I R^2 \left\{ \frac{1}{(R^2+(x+l)^2)^{3/2}} + \frac{1}{(R^2+(x-l)^2)^{3/2}} \right\}$$

(ii) $$\frac{\partial B_x}{\partial x} = 6\pi I R^2 \left\{ -\frac{l+x}{(R^2+(l+x)^2)^{5/2}} + \frac{(l-x)}{(R^2+(l-x)^2)^{5/2}} \right\}$$

at $x=l$

$$\frac{\partial B}{\partial x} = 6\pi I R^2 \left\{ \frac{-l+l}{(\;)^{5/2}} \right\} = 0.$$

iii)

$$\frac{\partial^2 B_x}{\partial x^2} = 2\pi R^2 \left\{ -\frac{3}{(R^2+(x-l)^2)^{5/2}} - \frac{3}{(R^2+(x+l)^2)^{5/2}} + \frac{15(l-x)^2}{(R^2+(l-x)^2)^{7/2}} + \frac{15(l+x)^2}{(R^2+(l+x)^2)^{7/2}} \right\}$$

Want $\frac{\partial^2 B}{\partial x^2} = 0$ at $x=0$

$$\frac{-6}{(R^2+l^2)^{5/2}} + \frac{\overset{30}{\cancel{15}}\,l^2}{(R^2+l^2)^{7/2}} = 0$$

$$R^2+l^2 = \frac{5}{\cancel{2}}\,l^2$$

$\div\, l^2$

$$\left(\frac{R}{l}\right)^2 = \frac{5}{2} - 1 = 4\ \frac{3}{2}$$

$$\boxed{\frac{R}{l} = 2\cancel{\sqrt{3/2}}}$$

I-3

D. Griffiths "Introduction to Electrodynamics"
Addison Wesley

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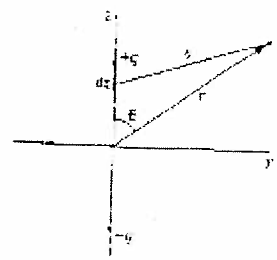


Figure 11.5

From the potentials, it is a straightforward matter to compute the fields.

$$\nabla V = \frac{\partial V}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial V}{\partial \theta}\hat{\boldsymbol{\theta}}$$

$$= -\frac{p_0\omega}{4\pi\epsilon_0 c}\left\{\cos\theta\left(-\frac{1}{r^2}\sin[\omega(t-r/c)] - \frac{\omega}{rc}\cos[\omega(t-r/c)]\right)\hat{\mathbf{r}} - \frac{\sin\theta}{r^2}\sin[\omega(t-r/c)]\hat{\boldsymbol{\theta}}\right\}$$

$$\cong \frac{p_0\omega^2}{4\pi\epsilon_0 c^2}\left(\frac{\cos\theta}{r}\right)\cos[\omega(t-r/c)]\hat{\mathbf{r}}.$$ $\rightarrow \frac{1}{r} \ll \frac{\omega}{c}$

(I dropped the first and last terms, in accordance with approximation 3.) Likewise,

$$\frac{\partial \mathbf{A}}{\partial t} = -\frac{\mu_0 p_0\omega^2}{4\pi r}\cos[\omega(t-r/c)](\cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}}),$$

and therefore

$$\boxed{\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t} = -\frac{\mu_0 p_0\omega^2}{4\pi}\left(\frac{\sin\theta}{r}\right)\cos[\omega(t-r/c)]\hat{\boldsymbol{\theta}}.} \quad (11.18)$$

Meanwhile

$$\nabla\times\mathbf{A} = \frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial\theta}\right]\hat{\boldsymbol{\phi}}$$

$$= -\frac{\mu_0 p_0\omega}{4\pi r}\left\{\frac{\omega}{c}\sin\theta\cos[\omega(t-r/c)] + \frac{\sin\theta}{r}\sin[\omega(t-r/c)]\right\}\hat{\boldsymbol{\phi}}.$$

The second term is again eliminated by approximation 3, so $\rightarrow \frac{1}{r} \ll \omega/c$

$$\boxed{\mathbf{B} = \nabla\times\mathbf{A} = -\frac{\mu_0 p_0\omega^2}{4\pi c}\left(\frac{\sin\theta}{r}\right)\cos[\omega(t-r/c)]\hat{\boldsymbol{\phi}}.} \quad (11.19)$$

Equations 11.18 and 11.19 represent monochromatic waves of frequency ω traveling in the radial direction at the speed of light. $\mathbf{E}$ and $\mathbf{B}$ are in phase, mutually perpendicular, and transverse; the ratio of their amplitudes is $E_0/B_0 = c$. All of which is precisely what we expect for electromagnetic waves in free space. (These are actually *spherical* waves, not plane waves, and their amplitude decreases like $1/r$ as they progress. But for large r, they are approximately plane over small regions—just as the surface of the earth is reasonably flat, locally.)

The energy radiated by an oscillating electric dipole is determined by the Poynting vector:

$$\mathbf{S} = \frac{1}{\mu_0}(\mathbf{E}\times\mathbf{B}) = \frac{\mu_0}{c}\left\{\frac{p_0\omega^2}{4\pi}\left(\frac{\sin\theta}{r}\right)\cos[\omega(t - r/c)]\right\}^2\hat{\mathbf{r}}. \tag{11.20}$$

The intensity is obtained by averaging (in time) over a complete cycle:

$$\langle \vec{S}\rangle = \frac{1}{T}\int_0^T \vec{S}\,dt$$

$$\langle\mathbf{S}\rangle = \left(\frac{\mu_0 p_0^2\omega^4}{32\pi^2 c}\right)\frac{\sin^2\theta}{r^2}\hat{\mathbf{r}}. \tag{11.21}$$

Notice that there is no radiation along the *axis* of the dipole (here $\sin\theta = 0$); the intensity profile[5] takes the form of a donut, with its maximum in the equatorial plane (Fig. 11.4). The total power radiated is found by integrating $\langle\mathbf{S}\rangle$ over a sphere of radius r:

$$\langle P\rangle = \int\langle\mathbf{S}\rangle\cdot d\mathbf{a} = \frac{\mu_0 p_0^2\omega^4}{32\pi^2 c}\int\frac{\sin^2\theta}{r^2}r^2\sin\theta\,d\theta\,d\phi = \frac{\mu_0 p_0^2\omega^4}{12\pi c}. \tag{11.22}$$

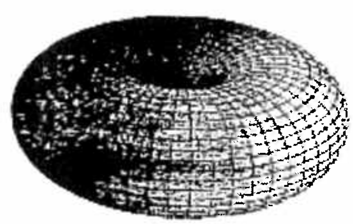


Figure 11.4

[5] The "radial" coordinate in Fig. 11.4 represents the magnitude of $\langle\mathbf{S}\rangle$ (at fixed r), as a function of θ and ϕ.

I- 10

Special relativity

Using the fact that the $k^\alpha = (\frac{w}{c}, \vec{k})$ is a four-vector, apply the Lorentz transformation to this quantity to derive the Doppler shift for a light wave, that is, the frequency observed in the laboratory for a source moving in the +x direction with velocity $v=\beta c$ for arbitrary **k**.

Solution: A four vector (x_0, **x**) transforms like

$x_0' = \gamma(x_0 - \boldsymbol{\beta}.\mathbf{x})$

$x'_{||} = \gamma (x_{||} - \beta x_0)$

$\mathbf{x'}_p = \mathbf{x}_p,$

where x_p is the component of x perpendicular to b. $\boldsymbol{\beta} = \mathbf{v}/c$, $\beta=|\boldsymbol{\beta}|$, and $\gamma = (1-\beta^2)^{-\frac{1}{2}}$. Given that $k^\alpha = \left(\frac{w}{c}, \vec{k}\right)$ is a four vector

$w'/c = \gamma(w/c - \boldsymbol{\beta}.\mathbf{k})$

This is the Doppler shift formula, $w' = \gamma w(1- \beta \cos(\theta))$, where $\cos(\theta)$ is the angle between **k** and $\boldsymbol{\beta}$, and $w=c|\mathbf{k}|$. The wavevector transforms like,

$k'_{||} = \gamma (k_{||} - \beta w/c)$

$\mathbf{k'}_p = \mathbf{k}_p,$

Physics PhD Qualifying Examination
Part II – Friday, January 22, 2010

Name:____
(please print)
Identification Number:________

STUDENT: insert a check mark in the left boxes to designate the problem numbers that you are handing in for grading.
PROCTOR: check off the right hand boxes corresponding to the problems received from each student. Initial in the right hand box.

	1	
	2	
	3	
	4	
	5	
	6	
	7	
	8	
	9	
	10	

Student's initials

problems handed in:

Proctor's initials

INSTRUCTIONS FOR SUBMITTING ANSWER SHEETS

1. DO NOT PUT YOUR NAME ON ANY ANSWER SHEET. EXAMS WILL BE COLLATED AND GRADED BY THE ID NUMBER ABOVE.
2. Use at least one separate preprinted answer sheet for each problem. Write on only one side of each answer sheet.
3. Write your **identification number** listed above, in the appropriate box on the preprinted sheets.
4. Write the **problem number** in the appropriate box of each preprinted answer sheet. If you use more than one page for an answer, then number the answer sheets with both problem number and page (e.g. Problem 9 – Page 1 of 3).
5. Staple together all the pages pertaining to a given problem. Use a paper clip to group together all eight problems that you are handing in.
6. Hand in a total of *eight* problems. A passing distribution will normally include at least four passed problems from problems 1-6 (Quantum Physics) and two problems from problems 7-10 (Thermodynamics and Statistical Mechanics). **DO NOT HAND IN MORE THAN EIGHT PROBLEMS.**
7. **YOU MUST SHOW ALL YOUR WORK.**

[II-1] **[10]**

In three dimensions, the Hamiltonian of a free particle is given by

$$H = -\frac{\hbar^2}{2m}\nabla^2$$

and its wavefunction

$$\psi_{\boldsymbol{k}}(\boldsymbol{r}) = C(\boldsymbol{k})e^{i\boldsymbol{k}\cdot\boldsymbol{r}}.$$

(a) Write down the allowed momenta or wave vectors for this system under the periodic boundary condition; i.e. upon translation by a fixed length L in the direction of any one of the three coordinate axes, all eigenfunctions of H shall assume their previous values.

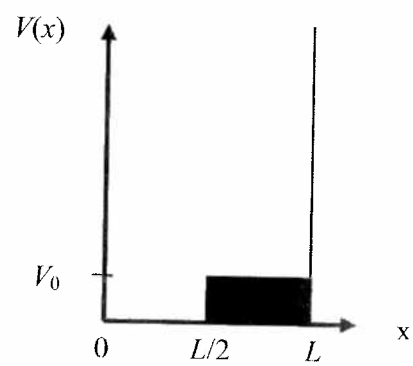
(b) What are the quantized energy values of this system?

(c) Calculate the energy density of state which is the number of eigenstates per unit energy interval.

[II-2] **[10]**

Consider a particle of mass m in an infinitely high potential well as shown in the figure. Suppose that the step V_0 at the bottom of the well can be considered as a small perturbation.

(a) Use first-order perturbation theory to calculate the eigenenergies E_n of the particle in the potential well.
(b) What are the first order corrected wavefunctions?
(c) If the particle is an electron, how do the frequencies emitted by the perturbed system compare with those of the unpertubed system?
(d) What smallness assumption is appropriate to V_0?



[II-3] **[10]**

The spin-orbit Hamiltonian is defined as

$$H = \xi \boldsymbol{L} \cdot \boldsymbol{S},$$

where ξ is a constant.

(a) Use perturbation theory using the atomic eigenfunctions of a one-electron atom (e.g., the Hydrogen atom) in the absence of spin-orbit interactions to find the eigenvalues of the state $|\psi\rangle$ in the presence of the spin-orbit coupling.

(b) Describe the degeneracy of the states. Work out the degeneracy for a state with $L = 1$ and $S = 1/2$.

[II-4] **[10]**

Suppose a potential that is described by

$$V(\boldsymbol{r}) = q^2 \ell^2 \,| R_{32}(r) Y_{20}(\theta) |^2 ,$$

where q is the electronic charge and ℓ is a length scale, and

$$R_{32}(r) = \frac{4}{81\sqrt{30}} a^{-3/2} (r/a)^2 e^{-r/(3a)} \;, \; Y_{20}(\theta) = \sqrt{\frac{5}{4\pi}} \left(\frac{3}{2}\cos^2(\theta) - \frac{1}{2} \right)$$

with a the Bohr radius. Let $\Delta = |\boldsymbol{k}' - \boldsymbol{k}|$ be the magnitude of the difference of the wave numbers for the scattered particle of mass m. For small Δa, obtain the differential cross section $d\sigma / d\Omega$ in the first order Born approximation, up to and including the term of order $(\Delta a)^2$.

[II-5] **[10]**

A pencil of mass $m = 10\,\text{g}$ and length $L = 15\,\text{cm}$ is balanced vertically on its point on a horizontal, completely rough surface. Neglecting all complicating effects, estimate the maximum length of time before the pencil makes an appreciable angle with the vertical. For concreteness, assume that $\varphi_1 \approx 0.1\,\text{rad}$ is an appreciable angle with the vertical.
<u>*Hint:*</u> Use the Heisenberg uncertainty principle expressed in a combination of an angular variable and angular momentum variable. You may also recall that the moment of inertia of a thin rod of length L and mass m, about an axis going through one of its end points, is $I = \frac{1}{3} mL^2$.

[II-6] **[10]**

Given the Hamiltonian of a classical radiation field (as opposed to quantized) below:

$$H = \frac{\boldsymbol{p}^2}{2m_e} + e\phi(\boldsymbol{x}) - \frac{e}{m_e c}\boldsymbol{A}\cdot\boldsymbol{p} \ ,$$

where the vector potential is

$$\boldsymbol{A}(\boldsymbol{x},t) = 2A_o\hat{\varepsilon}\cos\left(\frac{\omega}{c}\hat{\boldsymbol{n}}\cdot\boldsymbol{x} - \omega t\right)$$

for a monochromatic field.

(a) Write down the Fermi's golden rule.

(b) For the absorption case, estimate the transition rate from energy state $|\,i\rangle$ to state $|\,n\rangle$ of an atom excited by this electromagnetic wave. State $|\,n\rangle$ is a higher energy state relative to the state $|\,i\rangle$.

(c) Given the energy density of classical electromagnetic wave below,

$$u = \frac{1}{2}\left(\frac{E_{\max}^2}{8\pi} + \frac{B_{\max}^2}{8\pi}\right),$$

calculate the absorption cross section which is defined as

$$\frac{\textit{energy per quantum absorbed by the atom} \times \textit{transition rate}(i \to n)}{\textit{energy flux of the radiation field}} .$$

[II-7] **[7,3]**

A quantity of ideal monatomic gas consists of N atoms initially at temperature T_1. The pressure and volume are then slowly doubled in such a manner as to trace out a straight line on the $P-V$ diagram. **In terms of N, k, and T_1, find**

(a) the work W done by the gas.

(b) If one defines an equivalent average heat capacity ($C = Q/\Delta T$, where Q is the total heat transferred to the gas) for this particular process for the above monatomic ideal gas, what will its value be?

[II-8] **[10]**

A metal block of mass m and specific heat c with temperature T_b is placed into the ocean which has temperature T. Assume that the block and the ocean forms a closed system (isolated from the rest of the universe).

(a) What is the total change in entropy ΔS_{total} for the system (the block and the ocean combined)? You must express your answer in terms of m, c, T_b, and T. (The specific heat c of the block is assumed to be constant.)

(b) Show (*mathematically*) that your result in part **(a)** is consistent with the 2nd Law of Thermodynamics.

[II-9] **[10]**

The concentration of potassium K^+ ions in the internal sap of a plant cell (the fluid inside the plant cell) exceeds by a factor of 10^4 the concentration of K^+ ions in the pond water in which the cell is growing. The chemical potential of the K^+ ions is higher in the sap because their concentration n is higher there. Estimate the difference in the chemical potentials at $T = 300\,\mathrm{K}$ and determine the voltage across the cell wall. Take μ as the chemical potential for an ideal gas.

[II-10] **[2,2,6]**

Assume that the electrons in a metal can be described by a non-interacting ideal Fermi gas. Assume that we have a system of N free electrons (spin 1/2) enclosed in a volume V. Assume that the electrons are classical and do not interact via a potential, and $T = 0$.

(a) Plot the Fermi-Dirac distribution function for this system. Identify and define Fermi energy.
(b) Obtain the Fermi momentum and Fermi energy in terms of the number of electrons in the system and the volume.
(c) Obtain the pressure of the system in terms of the number of electrons in the system and the volume.

Solutions Part II

(II-1) ①/2

ⓐ Under the periodic boundary conditions, the allowed momenta are

$$\therefore \begin{cases} k_x = n_x \frac{2\pi}{L} & , n_x = 0, 1, 2, \text{—} \\ k_y = n_y \frac{2\pi}{L} & n_y = 0, 1, 2, \text{—} \\ k_z = n_z \frac{2\pi}{L} & n_z = 0, 1, 2, \text{—} \end{cases}$$

#

ⓑ $E = \frac{\hbar^2}{2m}(k_x^2 + k_y^2 + k_z^2)$

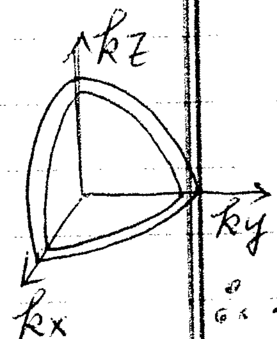
$$\therefore E = \frac{\hbar^2 \cdot 2\pi^2}{m L^2}(n_x^2 + n_y^2 + n_z^2)$$

#

(Ⅱ-1) ②/2

(c)

1 state: $(\Delta k)^3 = \left(\frac{2\pi}{L}\right)^3$ ①



$E \to E + \Delta E$

$k \to k + \Delta k \Rightarrow$ k-Volume: $4\pi k^2 \Delta k$ ②

∴ ΔN of allowed states from $k \to k + \Delta k$ is given by

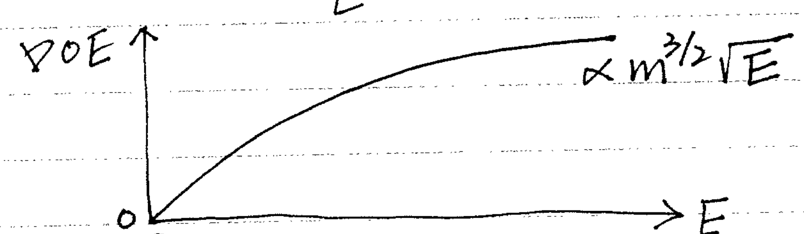
$$\Delta N = \frac{4\pi k^2 \Delta k}{\left(\frac{2\pi}{L}\right)^3} \quad ③$$

Given $E = \frac{\hbar^2 k^2}{2m}$, ∴ $\left.\begin{array}{l} \Delta E = \frac{\hbar^2 k \Delta k}{m} \\ k = \sqrt{2mE}\,\frac{1}{\hbar} \end{array}\right\}$ ④

Combine ③ & ④:

$$\Delta N = \frac{4\pi\left(\sqrt{2mE}/\hbar\right)\left(m\,\Delta E/\hbar^2\right)}{\left(\frac{2\pi}{L}\right)^3}$$

$$∴ \ DOS \equiv \frac{\Delta N}{\Delta E} = \frac{m^{3/2}\sqrt{E}}{\sqrt{2}\,\hbar^3\,\pi^2} \cdot L^3$$ #



$\boxed{\text{II-2}}$ particle in a 1-dimensional potential well

$$E_n = n^2 E_1 \qquad E_1 = \frac{\hbar^2 \pi^2}{2mL^2} \qquad n = 1, 2, 3, \ldots$$

$$\varphi_n = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right)$$

first order correction

$$E_n = E_n^{(0)} + H_{nn}'$$

$$\varphi_n = \varphi_n^{(0)} + \sum_{i \neq n} \frac{H_{in}'}{E_n^{(0)} - E_i^{(0)}} \varphi_i^{(0)}$$

$$H_{nn}' = \langle \varphi_n | H' | \varphi_n \rangle \qquad H_{in}' = \langle \varphi_i^0 | H' | \varphi_n^0 \rangle$$

$$H' = V_0 \qquad \frac{L}{2} \leq x \leq L$$

$$H_{nn}' = \int_{L/2}^{L} \frac{2}{L} \sin^2\left(\frac{n\pi x}{L}\right) V_0 \, dx$$

$$= \frac{2V_0}{L} \int_{L/2}^{L} \sin^2\left(\frac{n\pi x}{L}\right) dx$$

$$\int \sin^2 y \, dy = \frac{1}{2}(y - \cos y \sin y)$$

$$H_{nn}' = \frac{V_0}{2}$$

$$H_{in}' = \int_{L/2}^{L} \frac{2}{L} V_0 \sin\frac{i\pi x}{L} \sin\frac{n\pi x}{L} \, dx$$

$$= \frac{2V_0}{L} \int_{L/2}^{L} \sin\frac{i\pi x}{L} \sin\frac{n\pi x}{L} \, dx$$

$$\int \sin ax \sin bx \, dx = \frac{\sin[(a-b)x]}{2(a-b)} - \frac{\sin[(a+b)x]}{2(a+b)}$$

$$H_{in}' = \frac{V_0}{\pi}\left[\frac{\sin(i-n)\pi/2}{i-n} + \frac{\sin(i+n)\pi/2}{i+n}\right]$$

II-2

frequencies $\Delta E = \hbar\omega$ $E^0 = n^2 E_1$

$$E^1 = n^2 E_1 + \frac{V_0}{2}$$

$$\omega = \frac{1}{\hbar} \left(E_n - E_\ell \right) = \frac{1}{\hbar} E_1 (n^2 - \ell^2)$$

perturbed frequencies are the same
as unperturbed frequencies

smallness assumption $V_0 \ll E_1$

II - 3

Qualifier problems:

QM:

The spin-orbit Hamiltonian is defined as

$$H = \frac{1}{2m_e c^2} \frac{1}{r} \frac{\partial V}{\partial r} \, L \cdot S = \xi \, L \cdot S$$

a. Use perturbation theory using the atomic eigenfunctions of a one electron atom (e.g., the Hydrogen atom) in absence of spin orbit interactions to find the eigenvalues of the state $|\psi>$ in presence of the spin-orbit coupling.
b. Describe the degeneracy of the states. Work out the degeneracy for a state with L=1 and S=1/2.

Solution:

a)

$$J = L + S$$

$$L \cdot S = \frac{1}{2}(J^2 - L^2 - S^2)$$

The unperturbed $|\psi>$ is an eigenstate of J^2, L^2 and S^2, therefore

$$L \cdot S \,|\Psi\rangle = \frac{1}{2}\hbar^2 \xi \,[\,J(J+1) - L(L+1) - S(S+1)]|\Psi\rangle$$

b) The degeneracy is the degeneracy of m_J = 2J+1. For L=1, S=1/2, J= 3/2 with degeneracy 4, and J=1/2 with degeneracy 2. The J=3/2 have different eigenvalues than the J=1/2.

Solution II.4 The scattering amplitude is given by

$$\begin{aligned} f &= -\frac{m}{2\pi\hbar^2}\int d^3\mathbf{r}\; e^{i(\mathbf{k}'-\mathbf{k})\cdot\mathbf{r}}V(\mathbf{r}) \\ &= -\frac{me^2\ell^2}{2\pi\hbar^2}2\pi\int_{-1}^{1}d\mu\int_0^{\infty}r^2dr\left(1+i\mu\Delta r-\mu^2\Delta^2r^2+\mathcal{O}(\Delta^3)\right)|R_{32}(r)Y_{20}(\theta)|^2 \end{aligned}$$

where $\mu=\cos\theta$. For the leading term one just has

$$2\pi\int_{-1}^{1}d\mu\int_0^{\infty}r^2dr|R_{32}(r)Y_{20}(\theta)|^2=1$$

since it is the unit normalized hydrogen orbital ψ_{320}. The term proportional to Δ vanishes since the integrand is odd in μ. The term proportional to Δ^2 is:

$$-\Delta^2 2\pi\int_{-1}^{1}d\mu\frac{5}{4\pi}\mu^2\left(\frac{3}{2}\cos^2\theta-\frac{1}{2}\right)^2\int_0^{\infty}r^4drR_{32}^2(r)=-66(\Delta a)^2$$

Thus

$$f=-\frac{me^2\ell^2}{2\pi\hbar^2}\left[1-66(\Delta a)^2+\mathcal{O}((\Delta a)^4)\right]$$

It follows that

$$d\sigma/d\Omega=\frac{m^2e^4\ell^4}{4\pi^2\hbar^4}\left[1-132(\Delta a)^2+\mathcal{O}((\Delta a)^4)\right]$$

QE. Ph.D. Jan. 2010

(II-5) Solution :

The relevant Heisenberg relation is the one, involving the angular momentum, M, around the point of the pencil, $\Delta M . \Delta\varphi \sim \hbar$, where φ is the angle the pencil makes with the vertical. If I is the moment of inertia of the pencil, $I = \frac{1}{3}\mu d^2$ (μ: mass of the pencil; d: length of the pencil), we have $M = I\dot{\varphi}$ and thus

$$\Delta\dot{\varphi} . \Delta\varphi \sim \hbar / I. \tag{1}$$

The equation of motion of the pencil is $I\ddot{\varphi} = \frac{1}{2}\mu g d \sin\varphi \sim \frac{1}{2}\mu g d\varphi$ (g: gravitational acceleration), where we have assumed φ to be small. Solving this equation of motion we find

$$\varphi(t) = \varphi(0) \cosh kt + [\dot{\varphi}(0)/k] \sinh kt \quad (k^2 = 3g/2d),$$

and we find that the time t_1 necessary to reach a value φ_1 is given by the relation (we have used the fact that $kt_1 \gg 1$ so that $\cosh kt_1 \sim \sinh kt_1 \sim e^{kt}$)

$$t_1 \sim \frac{1}{k} \ln \frac{k\varphi_1}{k\varphi(0) + \dot{\varphi}(0)}.$$

Using (1) with $\Delta\dot{\varphi} = \dot{\varphi}(0)$, $\Delta\varphi = \varphi(0)$ to maximise t_1, we find for $\varphi_1 \sim 0{\cdot}1$

$$(t_1)_{\max} \sim \frac{1}{k} \ln \frac{\varphi_1}{2} \sqrt{\left(\frac{k}{\hbar I}\right)} \sim 2{\cdot}7 \text{ sec.}$$

II-6)

(a) Fermi's golden rule

$$w_{i\to n} = \frac{2\pi}{\hbar} |V_{ni}|^2 \delta(E_n - E_i)$$

$$V_{ni} = \langle n | V | i \rangle$$

where V is the perturbation term

(b) $$\cos\left(\frac{\omega}{c}\hat{n}\cdot\vec{x} - \omega t\right) = \frac{1}{2}\left[e^{i\left(\frac{\omega}{c}\right)\hat{n}\cdot\vec{x} - i\omega t} + e^{-i\left(\frac{\omega}{c}\right)\hat{n}\cdot\vec{x} + i\omega t}\right]$$

$$\vec{A} = A_0 \hat{\varepsilon}\Big[\underbrace{e^{i\left(\frac{\omega}{c}\right)\hat{n}\cdot\vec{x} - i\omega t}}_{\text{responsible for absorption}} + \underbrace{e^{-i\frac{\omega}{c}\hat{n}\cdot\vec{x} + i\omega t}}_{\text{responsible for stimulated emission}}\Big]$$

For absorption case:

$$V_{ni} = -\frac{eA_0}{m_e c}\left(e^{i(\omega/c)\hat{n}\cdot\vec{x}}\ \hat{\varepsilon}\cdot\vec{p}\right)_{ni}$$

$$w_{i\to n} = \frac{2\pi}{\hbar}\frac{e^2}{m_e^2 c^2}|A_0|^2 \left|\langle n | e^{i\left(\frac{\omega}{c}\right)(\hat{n}\cdot\vec{x})}\hat{\varepsilon}\cdot\vec{p} | i \rangle\right|^2 \delta(E_n - E_i - \hbar\omega)$$

(c) $$u = \frac{1}{2}\left(\frac{E_{max}^2}{8\pi} + \frac{B^2_{max}}{8\pi}\right)$$

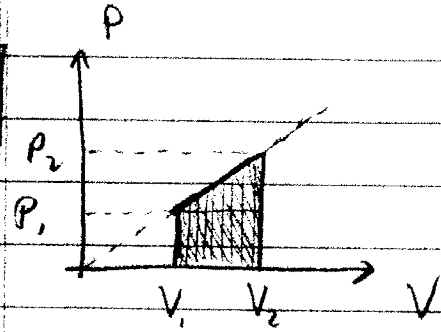
$$\text{energy flux} = cu = \frac{1}{2\pi}\frac{\omega^2}{c}|A_0|^2$$

(use $E = -\frac{1}{c}\frac{\partial}{\partial t}\vec{A}$ and $\vec{B} = \nabla\times\vec{A}$)

$$\sigma_{abs} = \frac{\hbar\omega\left(\frac{2\pi}{\cancel{\hbar}}\right)\left(\frac{\cancel{e^2}}{m_e^2 c^2}\right)|A_0|^2\left|\langle n | e^{i\left(\frac{\omega}{c}\right)(\hat{n}\cdot\vec{x})}\hat{\varepsilon}\cdot\vec{p} | i \rangle\right|^2 \delta(E_n - E_i - \hbar\omega)}{\left(\frac{1}{2\pi}\right)\left(\omega^2/c\right)|A_0|^2}$$

$$= \frac{4\pi^2\hbar}{2\omega}\left(\frac{e^2}{\hbar c}\right)\left|\langle n | e^{i\left(\frac{\omega}{c}\right)(\hat{n}\cdot\vec{x})}\hat{\varepsilon}\cdot\vec{p} | i \rangle\right|^2 \delta(E_n - E_i - \hbar\omega)$$

$\boxed{II-7}$



$P_2 = 2P_1$
$V_2 = 2V_1$

Along this line: $\frac{\Delta P}{\Delta V} = 2$ $\qquad \frac{P}{V} = \frac{P_1}{V_1}$

$$P(V) = \frac{P_1}{V_1} V$$

Further, $\left.\begin{array}{l} P_1 V_1 = NkT_1 \\ P_2 V_2 = NkT_2 \end{array}\right\}$ $\frac{T_2}{T_1} = \frac{P_2 V_2}{P_1 V_1} = 4 \Rightarrow T_2 = 4T_1$

a) $$W = \int_{V_1}^{V_2} P\,dV = \int_{V_1}^{V_2} \frac{P_1}{V_1} V\,dV = \frac{P_1}{V_1}\int_{V_1}^{V_2} V\,dV = \frac{P_1}{V_1}\frac{1}{2}\left(V_2^2 - V_1^2\right) = \frac{P_1}{2V_1}\left[(2V_1)^2 - V_1^2\right]$$

$$= \frac{3}{2} P_1 V_1 = \underline{\underline{\frac{3}{2} NkT_1}}$$ (work done by the gas)

Or, alternatively (using elementary geometry for the area):

$$W = \int_{V_1}^{V_1} P\,dV = \text{area under linear segment} =$$

$$= P_1(V_2 - V_1) + \frac{1}{2}(P_2 - P_1)(V_2 - V_1) = P_1 V_1 + \frac{1}{2} P_1 V_1 = \frac{3}{2} P_1 V_1$$

b)

$$\Delta U = Q - W$$ (W: work done by the gas)

$$Q = \Delta U + W$$

need change in internal energy:

ideal monatomic gas: $U = \frac{3}{2} NkT$

$$\Delta U = \frac{3}{2} Nk \Delta T = \frac{3}{2} Nk (T_2 - T_1) = \frac{3}{2} Nk\, 3T_1 = \frac{9}{2} NkT_1$$

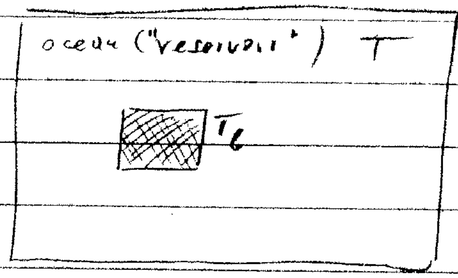
$$Q = \Delta U + W = \frac{9}{2} NkT_1 + \frac{3}{2} NkT_1 = \frac{12}{2} NkT_1 = 6NkT_1$$

Finally,

$$C \equiv \frac{Q}{\Delta T} = \frac{6NkT_1}{3T_1} = \underline{\underline{2Nk}}$$

(average heat capacity during this process)

II-8



m, c, T_b

The ocean works a reservoir at temperature T.

a) eventually the block will reach the temperature of the reservoir

$$\delta Q = mc\,dT \qquad Q = mc(T - T_b)$$

heat received by the block (heat given up by the reservoir

$$dS_{block} = \frac{\delta Q}{T'}$$

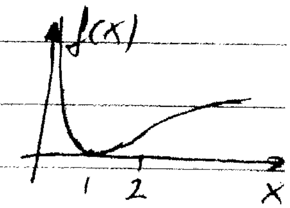
$$\Delta S_{block} = \int_{T_b}^{T} \frac{\delta Q}{T'} = \int_{T_b}^{T} \frac{mc\,dT'}{T'} = mc \ln\left(T/T_b\right)$$

$$\Delta S_{ocean} = \frac{-Q}{T} = -\frac{mc(T-T_b)}{T} = mc\left(\frac{T_b}{T} - 1\right)$$

$$\Delta S_{total} = \Delta S_{block} + \Delta S_{ocean} = \boxed{mc \ln\left(T/T_b\right) + mc\left(\frac{T_b}{T} - 1\right)}$$

b) $x \equiv T/T_b \qquad \Delta S_{total} = mc\left\{\underbrace{\ln(x) + \frac{1}{x} - 1}_{\equiv f(x)}\right\}$

$$f(x) = \ln(x) + 1/x - 1$$



$$f'(x) = \frac{1}{x} - \frac{1}{x^2} = \frac{x-1}{x^2} = \begin{cases} <0 & 0<x<1 \\ =0 & x=1 \\ >0 & x>1 \end{cases}$$

$f(x) = \min$ @ $x = 1$ and $f_{min} = f(1) = 0 \Rightarrow \boxed{\Delta S_{tot} \geq 0}$

Solution II-9 The difference in chemical potentials is simply given by

$$\Delta\mu = \mu(\text{sap}) - \mu(\text{pond}) = k_B T \ln n(\text{sap}) - k_B T \ln n(\text{sap}) = k_B T \ln 10^4 = 0.238 \text{ eV}$$

Thus the voltage difference is 0.238 V.

II-10

Quantum statistical mechanics

Ideal Fermi Gas at T=0.

Assume that the electrons in a metal can be described by a non-interacting ideal Fermi gas. Assume that we have system of N free electrons (spin 1/2) enclosed in a volume V. Assume that the electrons are classical and do not interact via a potential.

i) (2 PTS) Plot the Fermi-Dirac distribution function for this system. Identify and define Fermi energy.

ii) (3 PTS) Define the Fermi momentum and Fermi energy in terms of the number of electrons in the system and the volume.

iii) (5 pts) Show that the pressure of the system is given by

$$P = \frac{2}{3}\left(\frac{E}{V}\right) = (\frac{3h^3}{8\pi})^{2/3}\frac{n^{5/3}}{5m_e}$$

where n=N/V, m_e is the mass of the electron, and h is Planck' s constant.

Solution:

$$N = \frac{(2S+1)V}{h^3}\int_{p<p_F} d^3p = \frac{2V}{h^3}\frac{4}{3}\pi p_F^3$$

$$p_F = \left(\frac{3h^3}{8\pi}\right)^{1/3}\frac{N}{V}$$

$$E_F = \frac{p_F^2}{2m_e}$$

These equations define the Fermi momentum and Fermi energy in terms of n=N/V.

d) The average energy is given by

$$E = \frac{(2S+1)V}{h^3}\int_{p<p_F}\left(\frac{p^2}{2m_e}\right)d^3p = \frac{(2S+1)V}{h^3}\frac{4\pi}{5}\frac{p_F^5}{2m_e}$$

$$P = -\frac{\partial E}{\partial V} = \frac{2}{3}\frac{E}{V}\left(\frac{3h^3}{8\pi}\right)^{2/3}\frac{n^{5/3}}{5m_e}$$