Solution

Physics PhD Qualifying Examination
Part I – Wednesday, January 7, 2009

Name:____
(please print)
Identification Number:________

STUDENT: Designate the problem numbers that you are handing in for grading in the appropriate left hand boxes below. Initial the right hand box.
PROCTOR: Check off the right hand boxes corresponding to the problems received from each student. Initial in the right hand box.

	1	
	2	
	3	
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	8	
	9	
	10	

Student's initials

\# problems handed in:

Proctor's initials

INSTRUCTIONS FOR SUBMITTING ANSWER SHEETS

1. DO NOT PUT YOUR NAME ON ANY ANSWER SHEET. EXAMS WILL BE COLLATED AND GRADED BY THE ID NUMBER ABOVE.
2. Use at least one separate preprinted answer sheet for each problem. Write on only one side of each answer sheet.
3. Write your **identification number** listed above, in the appropriate box on each preprinted answer sheet.
4. Write the **problem number** in the appropriate box of each preprinted answer sheet. If you use more than one page for an answer, then number the answer sheets with both problem number and page (e.g. Problem 9 – Page 1 of 3).
5. Staple together all the pages pertaining to a given problem. Use a paper clip to group together all eight problems that you are handing in.
6. Hand in a total of *eight* problems. A passing distribution will normally include at least three passed problems from problems 1-5 (Mechanics) and three problems from problems 6-10 (Electricity and Magnetism). **DO NOT HAND IN MORE THAN EIGHT PROBLEMS.**
7. **YOU MUST SHOW ALL YOUR WORK.**

[I-1] **[2,1,2,3,2]**

A projectile is fired from height $y(0)=0$ and horizontal position $x(0)=0$ with an initial velocity $v_x(0)=v_{xo}$ and $v_y(0)=v_{yo}$. The force of air resistance is proportional to the momentum, $\boldsymbol{F}^{res}=-k\boldsymbol{p}$, where k is constant. The gravitational acceleration is g, pointing vertically down.

(a) Determine the velocity and position of the projectile as functions of time. That is, find $v_x(t), v_y(t), x(t), y(t)$ given the initial conditions described above.

(b) Find an implicit equation for the time T for the projectile to fall back to ground, $y(T)=0$.

(c) Thus, find the range $R=x(T)$. Find a way to express R as a simple linear function of T.

(d) Assuming that $kT \ll 1$ and $kv_{yo}/g \ll 1$, find an expression for T, including the first non-trivial correction,

$$T=c_1[1+c_2(kv_{yo}/g)+O(kv_{yo}/g)^2],$$

with c_1 and c_2 expressed in terms known constants and the initial conditions. Neglect the $O(kv_{yo}/g)^2$ term.

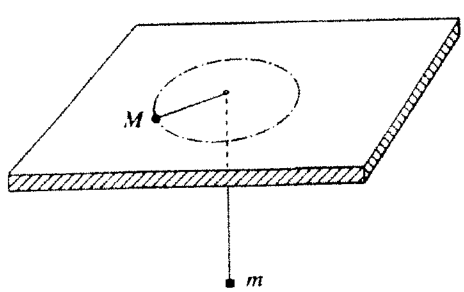
(e) Find a similar expression for R up to leading order in $O(kv_{yo}/g)$.

[I-2] **[6,4]**

A particle of mass M is constrained to move on a horizontal plane. A second particle, of mass m, is constrained to a vertical line. The two particles are connected by a mass-less string which passes through a hole in the plane (see Figure below). The motion is frictionless.

(a) Find the Lagrangian of the system and derive the equations of motion.

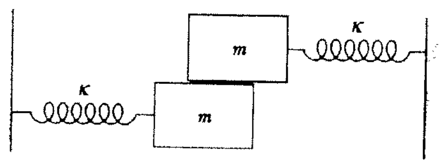
(b) Show that the orbit is stable with respect to small changes in the radius and find the frequency of small oscillations.



[I-3]   [10]

Two identical harmonic oscillators are placed such that the two masses $m_1=m_2=m$ slide against each other while oscillating. Consequently, a frictional force proportional to the instantaneous *relative* velocity, $F_i^{friction} = -b(\dot{x}_i - \dot{x}_j)$, acts on the sliding masses. (The motion of the two masses is horizontal along the x direction.)

(a) Find the equation of motion for each mass using Newtonian Mechanics.
(b) Solve for all possible solutions $x_1(t)$ and $x_2(t)$, depending on the values of κ, b and m.
(c) Discuss the motion of the two masses.
(You may express your answers in terms of $\omega_o \equiv \sqrt{\kappa/m}$ and the damping factor $\beta \equiv b/m$.)



[I-4]   [10]

A particle moves in a plane under the influence of a central force with a fixed force center. The mass of the particle is m, its angular momentum is l. The observed trajectory of the particle (expressed in planar polar coordinates) is given by

$$r(\vartheta) = ke^{\alpha\vartheta},$$

where k and α are constants. Find the force $\vec{F}(r)$ causing this motion.

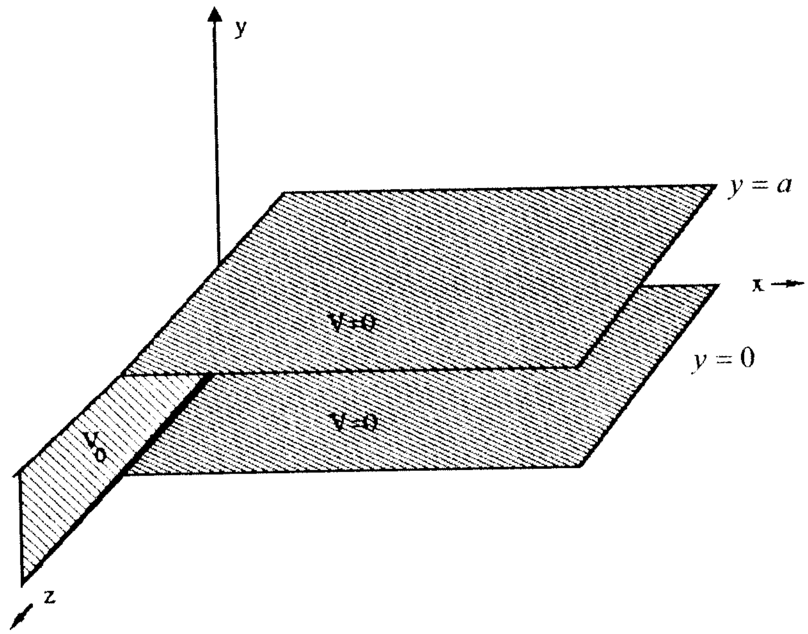
[I-5]   [1,1,1,1,6]

Consider a relativistic particle of mass m moving with velocity $\boldsymbol{v}$. Write down the following relations:

(a) Relativistic momentum $\boldsymbol{p}$ of a particle of mass m moving with velocity $\boldsymbol{v}$.
(b) Relativistic kinetic energy K of a particle of mass m moving with speed v.
(c) Rest energy E_0 of a particle of mass m.
(d) Total energy E of a particle of mass m moving with speed v.
(e) Show that the relativistic momentum and relativistic kinetic energy are related by: $p^2c^2 = 2Kmc^2 + K^2$ with speed of light $c = 299{,}792{,}468\,\text{m/s}$.

[I-6] **[10]**

Two semi-infinite *grounded* metal plates lie parallel to the xz plane, $x > 0$, one at $y = 0$, the other at $y = a$ as shown below. The left end at $x = 0$ is closed off with an infinite strip insulated from the two metal plates and maintained at a *constant* potential V_o. Find the potential *inside* this slot of width a, i.e., inside the region $x > 0$, $0 < y < a$. You may express your final answer as an infinite series, but you must determine all coefficients.



[I-7] **[10]**

Can the following vector functions represent static electric fields? If yes, determine the charge density.

(a) $\vec{E}(\vec{r}) = \vec{r} \times (\vec{c} \times \vec{r})$ ($\vec{c}$ is a constant vector);

(b) $\vec{E}(\vec{r}) = cr\vec{r}$ (c is a constant and $r = |\vec{r}|$).

Note: This is not a "yes or no" question; without showing the correct technical steps, you will get zero credit. This problem is to test your technical ability with differential vector operators. If you cannot demonstrate that you are competent in manipulating with various differential operators (related to $\vec{\nabla}$), you are not going to pass this problem.

[I-8] **[5,5]**

A long straight wire of radius b carries a current I in response to a voltage V between the ends of the wire.

(a) Calculate the Poynting vector $\vec{S}$ inside the wire ($r \le b$) for this DC voltage.
(b) Obtain the energy flux per unit length at the surface of the wire. Compare this result with Joule heating of the wire and comment on the physical significance.

[I-9] **[10]**

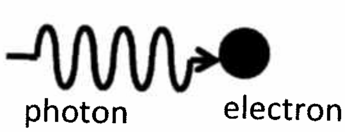
An electron is released from rest and falls under the influence of gravity. While falling a distance h, what fraction of the potential energy lost by the electron is radiated away?

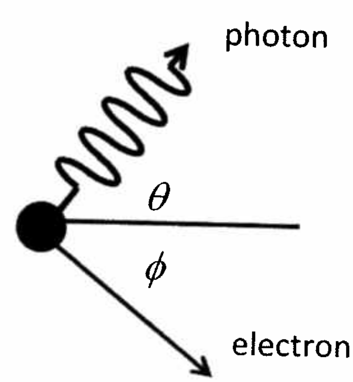
[I-10] **[10]**

Consider the scattering of a photon by an electron (Compton scattering). The electron is initially at rest. In the process, the photon losses some of its energy depending on the scattering angle ϑ,

$$\lambda = \lambda_o + \frac{h}{mc}[1 - \cos(\vartheta)]. \tag{1}$$

Here, λ is the wavelength of scattered photon. λ_o is the wavelength of incident photon. h is the Planck constant, m is the rest mass of electron, and c is the speed of light.
(a) Derive equation (1).
(b) Explain why the photon cannot be absorbed totally by the electron.





I-1

(a)

$$m\ddot{x} = -km\dot{x}\ , \qquad m\ddot{y} = -km\dot{y} - mg$$

$$\cancel{m}\dot{v}_x = -k\cancel{m}v_x \qquad \frac{dv_x}{v_x} = -k\,dt$$

$$\ln\frac{v_x(t)}{v_x(0)} = -kt$$

$$\boxed{v_x(t) = v_x(0)\,e^{-kt}}$$

$$\int_0^{x(t)} dx = v_x(0)\int_0^t dt\,e^{-kt} = v_x(0)\,\frac{1}{k}\,e^{-kt}\Big|_t^0$$

$$\boxed{x(t) = \frac{v_x(0)}{k}\left(1 - e^{-kt}\right)}$$

$$\dot{v}_y = -kv_y - g$$

We know homogeneous ($g=0$) part.
Add particular solution:

$$v_y = -\frac{g}{k}\,.$$

Thus

$$v_y(t) = c\,e^{-kt} - \frac{g}{k}$$

$$v_y(0) = c - \frac{g}{k}\ , \qquad c = \frac{g}{k} + v_y(0)$$

$$\boxed{v_y(t) = -\frac{g}{k} + \left(\frac{g}{k} + v_y(0)\right)e^{-kt}}$$

$$y(t) = -\frac{gt}{k} + \left(\frac{g}{k} + v_y(0)\right)\frac{1}{(-k)}\left(e^{-kt} - 1\right)$$

$$\boxed{y(t) = -\frac{gt}{k} + \left(\frac{g}{k} + v_y(0)\right)\frac{1}{k}\left(1 - e^{-kt}\right)}$$

(b) $y(T) = 0 \implies$

$$\frac{gT}{k} = \frac{g + kv_y(0)}{k^2}\left(1 - e^{-kT}\right)$$

$$\boxed{T = \frac{g + kv_y(0)}{gk}\left(1 - e^{-kT}\right)} \qquad (*)$$

(c) $R = x(T) = \frac{v_x(0)}{k}\left(1 - e^{-kT}\right)$

From $(*)$, $\left(1 - e^{-kT}\right) = \frac{kgT}{g + kv_y(0)}$.

$$\therefore \boxed{R = \frac{v_x(0)\, gT}{g + kv_y(0)}}$$

(d) Again from $(*)$,

$$T \approx \frac{g + kV_y(0)}{gk}\left(1 - \left(1 - kT + \frac{1}{2}(kT)^2 - \frac{1}{6}(kT)^3\right)\right)$$

$$= \frac{g + kV_y(0)}{kg}\left(kT - \frac{1}{2}(kT)^2 + \frac{1}{6}(kT)^3\right)$$

$$1 \approx \left(\frac{g + kV_y(0)}{g}\right)\left(1 - \frac{1}{2}(kT) + \frac{1}{6}(kT)^2\right)$$

$$\frac{g}{g + kV_y(0)} \approx 1 - \frac{1}{2}(kT) + \frac{1}{6}(kT)^2$$

$$\frac{1}{6}(kT)^2 - \frac{1}{2}(kT) + \frac{\cancel{g} + kV_y(0) - \cancel{g}}{g + kV_y(0)} \approx 0$$

$$(kT)^2 - 3(kT) + \frac{6kV_y(0)}{kV_y(0) + g} \approx 0$$

$$kT \approx \frac{3}{2} \pm \left[\left(\frac{3}{2}\right)^2 - \frac{6kV_y(0)}{kV_y(0) + g}\right]^{1/2}$$

must choose (−) for $kT \ll 1$, consistency.

$$kT \approx \frac{3}{2} - \frac{3}{2}\left[1 - \left(\frac{2}{3}\right)^2 \frac{6kV_y(0)/g}{1 + kV_y(0)/g}\right]^{1/2}$$

$$\approx \frac{2}{3} \cdot \frac{1}{2} \cdot \frac{6kV_y(0)/g}{1 + kV_y(0)/g} + \left(\frac{2}{3}\right)^3 \frac{1}{8}\left(\frac{6kV_y(0)}{g}\right)^2$$

$$\approx \frac{2kv_y(0)}{g}\left(1-\frac{kv_y(0)}{g}\right)+\frac{4}{3}\left(\frac{kv_y(0)}{g}\right)^2$$

$$\boxed{\therefore\ T=\frac{2v_y(0)}{g}\left(1-\frac{kv_y(0)}{3g}+\mathcal{O}\left(\frac{kv_y(0)}{g}\right)^2\right)}$$

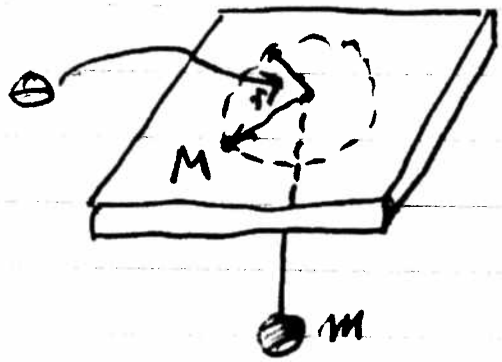
(e) $$R=\frac{v_x(0)\,T}{1+kv_y(0)/g}\approx\left(1-\frac{kv_y(0)}{g}\right)v_x(0)\,T$$

$$\approx v_x(0)\,\frac{2v_y(0)}{g}\left(1-\frac{4kv_y(0)}{3g}\right)$$

$$\boxed{\therefore\ R=\frac{2v_x(0)v_y(0)}{g}\left(1-\frac{4kv_y(0)}{3g}+\mathcal{O}\left(\frac{kv_y(0)}{g}\right)^2\right)}$$

(I-2) Solution

(a) We can write the Lagrangian as the length r of the string and the angle θ.



$$\mathcal{L} = \frac{1}{2}M(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{1}{2}m\dot{r}^2 - mgr$$

hence the equations of motion are:

$$(M+m)\ddot{r} - Mr\dot{\theta}^2 + mg = 0$$

$$\frac{d}{dt}(Mr^2\dot{\theta}) = 0$$

angular momentum is conserved hence,

$$Mr^2\dot{\theta} = \text{const.} = l_0$$

$$\therefore \dot{\theta} = l_0/Mr^2$$

$$\mathcal{L} = \frac{1}{2}(M+m)\dot{r}^2 + \frac{l_0^2}{2Mr^2} - mgr$$

(b) The equilibrium position is defined by taking the derivative of U_{eff} where

$$U_{eff} = mgr + \frac{l_0^2}{2Mr^2}$$

(I-2) <u>continued.</u>

$$\left.\frac{\partial U_{eff}}{\partial r}\right|_{r=r_0} = 0\,, \quad r_0 = \left(\frac{l_0^2}{gMm}\right)^{1/3}$$

$\frac{\partial^2 U_{eff}}{\partial r^2} > 0$, so the orbit is stable with respect to a small perturbation in the radius. The frequency of small oscillations is given by:

$$\omega^2 = \frac{1}{M_{eff}}\left(\frac{\partial^2 U_{eff}}{\partial r^2}\right)\Bigg|_{r=r_0} = \frac{1}{M+m}\left(\frac{\partial^2 U_{eff}}{\partial r^2}\right)\Bigg|_{r=}$$

$$= \frac{1}{M+m}\left(\frac{3 l_0^2}{M r_0^4}\right) = \frac{1}{1+\left(\frac{M}{m}\right)}\left(\frac{3g}{r_0}\right).$$

(1-3) Normal mode

1/3

Let x_1, x_2 be the displacements from their respective equilibrium positions.

The equ. of motion:

$$m\ddot{x}_1 + kx_1 + b(\dot{x}_1 - \dot{x}_2) = 0 \quad \cdots\cdots \quad (1)$$
$$m\ddot{x}_2 + kx_2 + b(\dot{x}_2 - \dot{x}_1) = 0 \quad \cdots\cdots \quad (2)$$

Let's combine equ. (1) & (2):

$(1)+(2) \Rightarrow \quad m(\ddot{x}_1 + \ddot{x}_2) + k(x_1 + x_2) = 0$

$(1)-(2) \Rightarrow \quad m(\ddot{x}_1 - \ddot{x}_2) + k(x_1 - x_2) + 2b(\dot{x}_1 - \dot{x}_2) = 0$

Now let $\eta_2 \overset{def}{=} (x_1 + x_2)$ & $\eta_1 \overset{def}{=} (x_1 - x_2)$

We have: $m\ddot{\eta}_2 + k\eta_2 = 0$ (3)

$m\ddot{\eta}_1 + k\eta_1 + 2b\dot{\eta}_1 = 0$ (4)

Solution for η_2:

$\eta_2 = A\cos(\omega_0 t + \delta)$, $\omega_0^2 \overset{def}{=} \frac{k}{m}$ (5)

Resonant frequency.

Solution for η_1:

Let $\eta_1 = e^{\gamma t + \delta}$ (6)

Equ.-(4) becomes: $m\gamma^2 + 2b\gamma + k = 0$

$$\gamma = \frac{-b \pm \sqrt{b^2 - mk}}{m} = -\frac{b}{m} \pm \sqrt{\frac{b^2}{m^2} - \frac{k}{m}}$$

Let $\beta \overset{def}{=} \frac{b}{m}$ (the damping term). (7)

$$\gamma = -\beta \pm \sqrt{\beta^2 - \omega_0^2} \quad (8)$$

If $\beta^2 < \omega_0^2$, $\eta_1 = e^{-\beta t} \cdot \cos(\sqrt{\beta^2 - \omega_0^2}\, t + \delta)$ (9)

If $\beta^2 = \omega_0^2$, $\eta_1 = e^{-\beta t}$ (10)

If $\beta^2 > \omega_0^2$, $\eta_1 = e^{-\beta t} \cdot e^{\pm\sqrt{\beta^2 - \omega_0^2}\, t}$ (11)

#

I-4

$$L = \frac{1}{2}\mu \dot{r}^2 + \frac{1}{2}\mu r^2 \dot{\vartheta}^2 - U(r) \qquad F(r) = -\frac{\partial U}{\partial r}$$

($m = \mu$ for fixed force center)

angular mom

$$\frac{\partial L}{\partial r} = \frac{d}{dt}\frac{\partial L}{\partial \dot{r}} \qquad 0 = \frac{\partial L}{\partial \vartheta} = \frac{d}{dt}\frac{\partial L}{\partial \dot{\vartheta}} \Rightarrow \mu r^2 \dot{\vartheta} = \ell = \text{const}$$

$$\dot{\vartheta} = \frac{\ell}{m r^2}$$

$$\mu r \dot{\vartheta}^2 - \frac{\partial U}{\partial r} = \mu \ddot{r}$$

$$\boxed{\mu r \dot{\vartheta}^2 + F(r) = \mu \ddot{r}}$$

One can show (Thornton & Marion, Sec. 8.4) (see next 2 pages) that the resulting orbit satisfies the following equation

$$\boxed{\frac{d^2}{d\vartheta^2}\left(\frac{1}{r}\right) + \frac{1}{r} = -\frac{\mu}{\ell^2} r^2 F(r)}$$

$$r(\vartheta) = k e^{\alpha\vartheta} \qquad 1/r = 1/k \, e^{-\alpha\vartheta}$$

$$\frac{d}{d\vartheta}\left(\frac{1}{r}\right) = -\frac{\alpha}{k} e^{-\alpha\vartheta}$$

$$\frac{d^2}{d\vartheta^2}\left(\frac{1}{r}\right) = \frac{\alpha^2}{k} e^{-\alpha\vartheta} = \alpha^2 \frac{1}{r}$$

$$\alpha^2 \frac{1}{r} + \frac{1}{r} = -\frac{\mu}{\ell^2} r^2 F(r) \Longrightarrow \boxed{F(r) = -\frac{\ell^2}{\mu}(1+\alpha^2)\frac{1}{r^3}}$$

(fixed forced center $\mu = m$)

$$\bar{F}(r) = F(r)\hat{r} = F(r)\frac{\bar{r}}{r}$$

$n = 1$ is just that of the harmonic oscillator (see Chapter 3), and the case $n = -2$ is the important inverse-square-law force treated in Sections 8.6 and 8.7. These two cases, $n = 1, -2$, are of prime importance in physical situations. Details of some other cases of interest will be found in the problems at the end of this chapter.

We have therefore solved the problem in a formal way by combining the equations that express the conservation of energy and angular momentum into a single result, which gives the equation of the orbit $\theta = \theta(r)$. We can also attack the problem using Lagrange's equation for the coordinate r:

$$\frac{\partial L}{\partial r} - \frac{d}{dt}\frac{\partial L}{\partial \dot{r}} = 0$$

Using Equation 8.7 for L, we find

$$\mu(\ddot{r} - r\dot{\theta}^2) = -\frac{\partial U}{\partial r} = F(r) \tag{8.18}$$

Equation 8.18 can be cast in a form more suitable for certain types of calculations by making a simple change of variable:

$$u \equiv \frac{1}{r}$$

First, we compute

$$\frac{du}{d\theta} = -\frac{1}{r^2}\frac{dr}{d\theta} = -\frac{1}{r^2}\frac{dr}{dt}\frac{dt}{d\theta} = -\frac{1}{r^2}\frac{\dot{r}}{\dot{\theta}}$$

But from Equation 8.10, $\dot{\theta} = l/\mu r^2$, so

$$\frac{du}{d\theta} = -\frac{\mu}{l}\dot{r}$$

Next, we write

$$\frac{d^2u}{d\theta^2} = \frac{d}{d\theta}\left(-\frac{\mu}{l}\dot{r}\right) = \frac{dt}{d\theta}\frac{d}{dt}\left(-\frac{\mu}{l}\dot{r}\right) = -\frac{\mu}{l\dot{\theta}}\ddot{r}$$

and with the same substitution for $\dot{\theta}$, we have

$$\frac{d^2u}{d\theta^2} = -\frac{\mu^2}{l^2}r^2\ddot{r}$$

Therefore, solving for $\ddot{r}$ and $r\dot{\theta}^2$ in terms of u, we find

$$\left.\begin{aligned} \ddot{r} &= -\frac{l^2}{\mu^2}u^2\frac{d^2u}{d\theta^2} \\ r\dot{\theta}^2 &= \frac{l^2}{\mu^2}u^3 \end{aligned}\right\} \tag{8.19}$$

Substituting Equation 8.19 into Equation 8.18, we obtain the transformed equation of motion:

$$\frac{d^2u}{d\theta^2} + u = -\frac{\mu}{l^2}\frac{1}{u^2}F(1/u) \tag{8.20}$$

which we may also write as

$$\boxed{\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r} = -\frac{\mu r^2}{l^2}F(r)} \tag{8.21}$$

This form of the equation of motion is particularly useful if we wish to find the force law that gives a particular known orbit $r = r(\theta)$.

EXAMPLE 8.1

Find the force law for a central-force field that allows a particle to move in a logarithmic spiral orbit given by $r = ke^{\alpha\theta}$, where k and α are constants.

Solution. We use Equation 8.21 to determine the force law $F(r)$. First, we determine

$$\frac{d}{d\theta}\left(\frac{1}{r}\right) = \frac{d}{d\theta}\left(\frac{e^{-\alpha\theta}}{k}\right) = \frac{-\alpha e^{-\alpha\theta}}{k}$$

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) = \frac{\alpha^2 e^{-\alpha\theta}}{k} = \frac{\alpha^2}{r}$$

From Equation 8.21, we now determine $F(r)$.

$$F(r) = \frac{-l^2}{\mu r^2}\left(\frac{\alpha^2}{r} + \frac{1}{r}\right)$$

$$F(r) = \frac{-l^2}{\mu r^3}(\alpha^2 + 1) \tag{8.22}$$

Thus, the force law is an attractive inverse cube.

EXAMPLE 8.2

Determine $r(t)$ and $\theta(t)$ for the problem in Example 8.1.

Solution. From Equation 8.10, we find

$$\dot{\theta} = \frac{l}{\mu r^2} = \frac{l}{\mu k^2 e^{2\alpha\theta}} \tag{8.23}$$

Rearranging Equation 8.23 gives

$$e^{2\alpha\theta}d\theta = \frac{l}{\mu k^2}dt$$

I-5

(a) $\vec{p} = \gamma m \vec{v}$ $\qquad \gamma = \dfrac{1}{\sqrt{1 - v^2/c^2}}$

(b) $K = mc^2(\gamma - 1)$

(c) $E_0 = mc^2$

(d) $E = E_0 + K = \gamma mc^2$

(e) $2Kmc^2 + K^2 = 2mc^2(\gamma-1)mc^2 + m^2c^4(\gamma-1)^2$

$$
\begin{aligned}
&= 2m^2c^4(\gamma-1) + m^2c^4(\gamma-1)^2 \\
&= m^2c^4\left[2(\gamma-1) + (\gamma-1)^2\right] \\
&= m^2c^4\left[(\gamma-1)^2 + 2(\gamma-1) + 1 - 1\right] \\
&= m^2c^4\left((\gamma-1)+1\right)^2 - m^2c^4 \\
&= m^2c^4\gamma^2 - m^2c^4 \\
&= E^2 - E_0^2
\end{aligned}
$$

$$
\begin{aligned}
c^2p^2 &= \gamma^2m^2v^2c^2 \\
&= \gamma^2m^2c^4\left(\frac{v^2}{c^2}\right) \\
&= \gamma^2m^2c^4\left(1 - \frac{1}{\gamma^2}\right) = \gamma^2m^2c^4 - m^2c^4 \\
&= E^2 - E_0^2
\end{aligned}
$$

$\Downarrow \qquad c^2p^2 = 2Kmc^2 + K^2$

I-6 $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) V = 0$

$V = X(x)\,Y(y)$, separable

B.C.s : $V(y=0) = 0, \quad V(y=a) = 0$
$V(x=0) = V_0, \quad V(x\to\infty) = 0.$

$$\frac{1}{X}\frac{d^2X}{dx^2} = -\frac{1}{Y}\frac{d^2Y}{dy^2} = k$$

$$\Rightarrow \quad X(x) = A e^{kx} + B e^{-kx}$$

$$Y(y) = C \sin ky + D \cos ky$$

$V(x\to\infty) = 0 \Rightarrow A = 0.$

Absorb B into C, D. Then

$$V = e^{-kx}(C \sin ky + D \cos ky)$$

$V(y=0) = 0 \Rightarrow D = 0$

$V(y=a) = 0 \Rightarrow \sin ka = 0$

$\Rightarrow k = 0 \mod \frac{\pi}{a}$

but $k \leq 0$ ruled out by $V(x\to\infty) = 0.$

Thus $V = \sum_{n=1}^{\infty} c_n e^{-n\pi x/a} \sin \frac{n\pi y}{a}$.

$$V_0 = \sum_{n>0} c_n \sin \frac{\pi n y}{a}$$

$$\int_0^a dy \sin \frac{\pi n' y}{a} V_0 = \sum_{n>0} c_n \int_0^a \sin \frac{\pi n y}{a} \sin \frac{\pi n' y}{a} dy$$

$$= \sum_{n>0} c_n \frac{a}{2} \delta_{n,n'} = c_{n'} \frac{a}{2}$$

$$= -\frac{V_0 a}{\pi n'} \cos \frac{\pi n' y}{a} \Big|_0^a = \begin{cases} \frac{2V_0 a}{\pi n'} & (n' \text{ odd}) \\ 0 & (n' \text{ even}) \end{cases}$$

$$\boxed{\therefore V = \sum_{n=0}^{\infty} \frac{4V_0}{\pi(2n+1)} e^{-\frac{\pi x}{a}(2n+1)} \sin \frac{\pi y}{a}(2n+1)}$$

I-7 Can the following vector functions represent a static electric field? If yes, determine the charge density.

a) $\bar{E}(\bar{r}) = \bar{r} \times (\bar{c} \times \bar{r})$

b) $\bar{E}(\bar{r}) = c\, r\, \bar{r}$ $\qquad (r \equiv |\bar{r}|)$

---

a) $\bar{E}(\bar{r}) = \bar{r} \times (\bar{c} \times \bar{r}) = \bar{c}(\bar{r}\cdot\bar{r}) - \bar{r}(\bar{c}\cdot\bar{r}) = \bar{c}\, r^2 - \bar{r}(\bar{c}\cdot\bar{r})$

$\text{rot}\, \bar{E} = \nabla \times \bar{E} = \nabla \times \left(\bar{c}\, r^2 - \bar{r}(\bar{c}\cdot\bar{r})\right) =$

$= (\nabla r^2) \times \bar{c} - \nabla \times \left(\bar{r}(\bar{c}\cdot\bar{r})\right) =$

$\left(\nabla r = \frac{\bar{r}}{r}\right)$ $\quad = 2r(\nabla r) \times \bar{c} - \left[(\bar{c}\cdot\bar{r})\underbrace{(\nabla \times \bar{r})}_{=\emptyset} + (\nabla(\bar{c}\cdot\bar{r})) \times \bar{r}\right]$

$= 2\bar{r} \times \bar{c} - \bar{c} \times \bar{r} = 2\bar{r} \times \bar{c} + \bar{r} \times \bar{c} = 3\bar{r} \times \bar{c} \neq 0$

Thus, a) cannot represent static $E(\bar{r})$ field

I-7 cont.

b)

$$\bar{\nabla}\times\bar{E} = \bar{\nabla}\times(c\,r\,\bar{r}) = (\bar{\nabla}\cdot c\,r)\times\bar{r} + c\,r\cdot\underbrace{\bar{\nabla}\times\bar{r}}_{=\phi}$$

$$= c\,\frac{\bar{r}}{r}\times\bar{r} \quad \equiv 0$$

yes, b) can represent a static $\bar{E}(\bar{r})$ field

Then, $\rho(\bar{r}) = \varepsilon_0\,\bar{\nabla}\cdot\bar{E}$

$$\frac{\rho(\bar{r})}{\varepsilon_0} = \bar{\nabla}\cdot(c\,r\,\bar{r}) = c\,(\bar{\nabla}r)\cdot\bar{r} + c\,r\,(\bar{\nabla}\cdot\bar{r})$$

$$= c\,\frac{\bar{r}}{r}\cdot\bar{r} + c\,r\,3 \quad = c\,r + 3\,c\,r = 4\,c\,r$$

$$\boxed{\rho(\bar{r}) = \varepsilon_0\,4\,c\,r}$$

(I-8) Solution:

(a) Let us calculate the flux of the Poynting vector. Introduce cylindrical coordinates with unit vectors $\hat{e}_\rho$, $\hat{e}_\theta$ and $\hat{z}$. Current flows along the wire in the z-direction and the electric field $\vec{E} = E\hat{z}$. Using one of the Maxwell's equation in vacuum the fact that conditions are stationary and Stokes' theorem

$$\vec{\nabla}\times\vec{B} = \frac{4\pi}{c}\vec{J} + \frac{1}{c}\frac{\partial E}{\partial t}$$

$$\int_A \vec{\nabla}\times\vec{B}\cdot d\vec{A} = \oint_c \vec{B}\cdot d\vec{\ell} = \frac{4\pi}{c}\int_A \vec{J}\cdot d\vec{A}$$

here $\vec{J}$ is the current density and $\vec{A}$ the surface. At any given radius r, B_θ is constant, so we have

$$2\pi r B = \frac{4\pi}{c} J \pi r^2, \quad \vec{B} = \frac{2\pi}{c} J r \hat{e}_\theta$$

$$\vec{S} = \frac{c}{4\pi}\vec{E}\times\vec{B} = \frac{c}{4\pi}\frac{2\pi}{c} J r E(\hat{z}\times\hat{e}_\theta)$$

$$\vec{S} = \frac{1}{2} J r E \hat{e}_\rho$$

Using the relation between current density and total current

$$J = I/(\pi b^2):$$

(I-8) continued.

$$\vec{S} = -\frac{IEr}{2\pi b^2}\hat{e}_\rho\ , \quad \vec{S}(b) = -\frac{IE}{2\pi b}\hat{e}_\rho$$

(b) The Poynting flux per unit length is then $\vec{S}\cdot 2\pi b = -IE$. So the flux enters the wire, and we see that the dissipated power per unit length IE is equal to the total incoming S-flux in agreement with Poynting's theorem

$$\frac{\partial u}{\partial t} = -\vec{J}\cdot\vec{E} - \vec{\nabla}\cdot\vec{S}$$

where u is the energy density. Under stationary conditions as here $\frac{\partial u}{\partial t}=0$, and we have

$$\int_V \vec{J}\cdot\vec{E}\, d^3x = -\int_V \vec{\nabla}\cdot\vec{S}\, dV = -\int_A \vec{S}\cdot d\vec{A} = IE$$

1-1

Problem 11.10

$\mathbf{p} = -ey\,\hat{\mathbf{y}}$, $y = \frac{1}{2}gt^2$, so $\mathbf{p} = -\frac{1}{2}get^2\,\hat{\mathbf{y}}$; $\ddot{\mathbf{p}} = -ge\,\hat{\mathbf{y}}$. Therefore (Eq. 11.60) : $P = \frac{\mu_0}{6\pi c}(ge)^2$. Now, the time it takes to fall a distance h is given by $h = \frac{1}{2}gt^2 \Rightarrow t = \sqrt{2h/g}$, so the energy radiated in falling a distance h is $U_{\text{rad}} = Pt = \frac{\mu_0(ge)^2}{6\pi c}\sqrt{2h/g}$. Meanwhile, the potential energy lost is $U_{\text{pot}} = mgh$. So the *fraction* is

$$f = \frac{U_{\text{rad}}}{U_{\text{pot}}} = \frac{\mu_0 g^2 e^2}{6\pi c}\sqrt{\frac{2h}{g}}\frac{1}{mgh} = \boxed{\frac{\mu_0 e^2}{6\pi mc}\sqrt{\frac{2g}{h}}.} = \frac{(4\pi \times 10^{-7})(1.6 \times 10^{-19})^2}{6\pi(9.11 \times 10^{-31})(3 \times 10^{8})}\sqrt{\frac{(2)(9.8)}{(0.02)}} = \boxed{2.76 \times 10^{-22}.}$$

Evidently ***almost*** all the energy goes into kinetic form (as indeed I *assumed* in saying $y = \frac{1}{2}gt^2$).

① Conservation of momentum: $\vec{p_e} = \vec{P_1} - \vec{P_2}$

② conserva. of energy: $h\nu + m_0c^2 = h\nu' + (p_e^2c^2 + m_0^2c^4)^{1/2}$

from ①: $|\vec{p_e}|^2 = |\vec{p_1}|^2 + |\vec{p_2}|^2 - 2|\vec{p_1}||\vec{p_2}|\cos\theta$

$$|p_e|^2 = \left(\frac{h\nu}{c}\right)^2 + \left(\frac{h\nu'}{c}\right)^2 - 2\left(\frac{h\nu}{c}\right)\left(\frac{h\nu'}{c}\right)\cos\theta$$

$$\therefore p_e^2c^2 = (h\nu)^2 + (h\nu')^2 - 2(h\nu)(h\nu')\cos\theta \quad \text{------ ③}$$

(Note: for $\theta = 0$, $p_ec = h\nu - h\nu'$: forward scattering.)

from ②: $p_e^2c^2 + m_0^2c^4 = (h\nu - h\nu' + m_0c^2)^2$

$$\therefore p_e^2c^2 = (h\nu - h\nu')^2 + 2m_0c^2(h\nu - h\nu') \quad \text{---- ④}$$

(Note: for $\theta = 0$, $p_ec = h\nu - h\nu'$; use equ.④ $\Rightarrow h\nu - h\nu' = 0$.
i.e. at $\theta = 0$, no energy transfer, no momentum transfer

Combine equ-③ & equ.-④:

$$(h\nu)^2 + (h\nu')^2 - 2(h\nu)(h\nu')\cos\theta = (h\nu)^2 + (h\nu')^2 - 2(h\nu)(h\nu') + 2m_0c^2(h\nu - h\nu')$$

$$\therefore 2(h\nu)(h\nu')(1 - \cos\theta) = 2m_0c^2(h\nu - h\nu') \quad \text{...... ⑤}$$

$$\frac{h}{m_0c^2}(1 - \cos\theta) = \frac{1}{\nu} - \frac{1}{\nu'}$$

$$\therefore \frac{h}{m_0c}(1 - \cos\theta) = \lambda' - \lambda \quad \text{.... ⑥}$$

Compton Wavelength.

I-10

As. $\lambda' = \frac{h}{m_0 c}(1 - \cos\theta) + \lambda_0$

$$\lambda' \leq \lambda_0 + 2\frac{h}{m_0 c} \quad \cdots\cdots (7)$$

As "m_0" of an electron is finite,
λ' will always be a finite value.

It is only when "m_0" $\to 0$ that λ' can approach "∞" and therefore the incoming photon is "totally absorbed".

Solutions

Physics PhD Qualifying Examination
Part II – Friday, January 9, 2009

Name:____
(please print)
Identification Number:________

STUDENT: insert a check mark in the left boxes to designate the problem numbers that you are handing in for grading.
PROCTOR: check off the right hand boxes corresponding to the problems received from each student. Initial in the right hand box.

	1	
	2	
	3	
	4	
	5	
	6	
	7	
	8	
	9	
	10	

Student's initials

problems handed in:

Proctor's initials

INSTRUCTIONS FOR SUBMITTING ANSWER SHEETS

1. DO NOT PUT YOUR NAME ON ANY ANSWER SHEET. EXAMS WILL BE COLLATED AND GRADED BY THE ID NUMBER ABOVE.
2. Use at least one separate preprinted answer sheet for each problem. Write on only one side of each answer sheet.
3. Write your **identification number** listed above, in the appropriate box on the preprinted sheets.
4. Write the **problem number** in the appropriate box of each preprinted answer sheet. If you use more than one page for an answer, then number the answer sheets with both problem number and page (e.g. Problem 9 – Page 1 of 3).
5. Staple together all the pages pertaining to a given problem. Use a paper clip to group together all eight problems that you are handing in.
6. Hand in a total of *eight* problems. A passing distribution will normally include at least four passed problems from problems 1-6 (Quantum Physics) and two problems from problems 7-10 (Thermodynamics and Statistical Mechanics). **DO NOT HAND IN MORE THAN EIGHT PROBLEMS.**
7. **YOU MUST SHOW ALL YOUR WORK.**

[II-1] [10]

Consider a particle in a one-dimensional potential well of width L with infinitely high walls:

$$V(x) = \begin{cases} 0 & \text{if } 0 < x < L \\ \infty & \text{otherwise} \end{cases}.$$

At $t = 0$ the wave function is

$$\psi(x,0) = \begin{cases} \sqrt{\dfrac{2}{L}} & \text{for} \quad L/4 < x < 3L/4 \\ 0 & \text{otherwise} \end{cases}.$$

At time $t > 0$ we measure the energy of the particle. What is the probability that the energy of the particle is ***greater than*** the ground-state energy of the particle in the potential-well?

[II-2] [10]

Consider a harmonic oscillator, $H^o = \dfrac{p^2}{2m} + \dfrac{1}{2} m\omega_o^2 x^2$, with a stationary perturbation term,

$W = \dfrac{1}{2}\rho m\omega_o^2 x^2$ ($\rho << 1$).

(a) First, solve for the *exact* eigenvaules of $H = H^o + W$, and expand the eigenvalues in powers of ρ.

(b) Next, determine the energy levels of H up to first order corrections, by treating W as a *time-independent* perturbation.

(Hint: Express W in terms of $a = \sqrt{\dfrac{m\omega_o}{2\hbar}}\left(x + \dfrac{i}{m\omega_o}p\right)$ and $a^+ = \sqrt{\dfrac{m\omega_o}{2\hbar}}\left(x - \dfrac{i}{m\omega_o}p\right)$ (creation and annihilation operator for the harmonic oscillator) and work with the matrix elements $\langle \psi_m^o | W | \psi_n^o \rangle$, where ψ_n^o are the eigenstates of H^o).

[II-3] [1,2,2,2,2,1]

Consider an electron spin $\vec{S}$ in a time-dependent magnetic field

$$\vec{B} = B_o\hat{z} + B_1\hat{x}\cos(\omega t) + B_1\hat{y}\sin(\omega t).$$

The magnetic moment of the electron is given by $\vec{\mu} = -\frac{e}{m}\vec{S}$, where e is the magnitude of its charge and m is its mass. In this problem, we will only consider the interaction between the magnetic moment of the electron and the magnetic field.

(a) Express the Hamiltonian as a 2×2 matrix using the explicit form of $\vec{S}$.

(b) We will assume a solution of the time-dependent Schrödinger equation of the form

$$\psi(t) = e^{i\lambda t}\begin{pmatrix} a_1 e^{\;i\omega t/2} \\ a_2 e^{i\omega t/2} \end{pmatrix}.$$

Using the time-dependent Schrödinger equation, obtain a system of equations for a_1 and a_2.

(c) From the constraint of a non-trivial solution for a_1 and a_2, determine the two possible solutions $\lambda_\pm$ for λ. Express your answers in term of $\omega_o \equiv eB_o/m$ and $\omega_1 \equiv eB_1/m$.

(d) Let $\omega = \omega_o$. Then find a_1 and a_2, and the corresponding solutions $\psi_\pm(t)$ for the two allowed values $\lambda_\pm$ of λ.

(e) Suppose that the electron is in a spin-up state at $t = 0$. Find the corresponding $\psi(t)$ by taking the appropriate linear combination of the solutions in (d).

(f) Compute $\langle S_z \rangle$ as a function of time for the solution you found in (e)

[II-4] [5,5]

Evaluate in the Born approximation the cross-section for scattering by a "delta-function" potential. The scattering potential, or rather potential energy, is equal to $V(\mathbf{r}) = B\,\delta(\mathbf{r})$, where we take the force center as the origin. B is a constant and is clearly equal to the volume integral of the potential: $B = \int V(\mathbf{r})d^3r$ = constant.

(a) Obtain the differential scattering cross section and the total scattering cross section using the Born approximation.

(b) Discuss this case with a delta-function potential with respect to the interaction potential of very small range which is much less than the deBroglie wavelength. To what type of particles is this case applicable? Is this scattering isotropic and thus velocity independent? Discuss the applicability of the Born approximation.

[II-5] **[10]**

Given the potential energy of $V(r) = -\frac{1}{4\pi\varepsilon_o}\frac{e^2}{r}$, use the uncertainty principle, $\Delta p \Delta x \geq \hbar$ to find the Bohr radius r_o and the ground state energy E_o of a Hydrogen atom.
(Hint: write down the kinetic energy in terms of r_o using the uncertainty principle.)

[II-6] **[10]**

A one-dimensional harmonic oscillator is in its ground state ψ_o at $t = -\infty$. It is perturbed by a small time-dependent potential $V(t) = -\alpha\, x \exp(-t^2/\tau^2)$ [α and τ are constants, x is the position of the oscillator.]. What is the probability of finding the oscillator in the first excited state ψ_1 at $t = +\infty$? The following expressions may be helpful:

$$\psi_n(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} \frac{1}{\sqrt{2^n n!}} \exp\left(-\frac{m\omega}{2\hbar}x^2\right) H_n\left(x\sqrt{\frac{m\omega}{\hbar}}\right),$$

$$H_0(x) = 1,$$
$$H_1(x) = 2x.$$

[II-7] **[10]**

Consider a gas with the equation of state

$$P(T,n) = -an^2 - \frac{kT}{b}\log(1-bn),$$

where T is the absolute temperature, $n = N/V$ (number of particles per unit volume), a and b are positive material-specific parameters, and k is the Boltzman constant. The constant-volume heat capacity of the gas is a constant, $C_V > 0$.

Consider that this gas undergoes *"free expansion"* from V_1 to V_2. (In this process, also referred to as the Joule experiment, the gas is thermally insulated from its environment and "suddenly" expands into vacuum.) Its initial temperature is T_1. Obtain the final temperature T_2 of the gas. (You must express the final temperature T_2 in terms of T_1, V_1, V_2, and other constant given in the problem.)

[II-8] **[6,4]**

(a) Consider a system that is partially in the vapour phase and partially in the liquid phase such that one may consider it in a chemical equilibrium. Such a system is described to exhibit a first order phase transition as it undergoes a phase change from the vapour phase to the liquid phase. The equation that describes such a phase transition is called the "Clausius-Clapeyron equation". The Clausius-Clapeyron equation is a relationship between "observable quantities" and determines the co-existence line in the *P-T* plane, namely the "vapour-liquid curve". Derive this equation and show that it has the form:

$$\left(\frac{dP}{dT}\right)_{v-l} = \frac{L}{T(v_v - v_l)},$$

here L is the latent heat of transition, v_v and v_l are the volumes of the vapour and liquid phase, respectively and P and T denote pressure and temperature.

Hint: In equilibrium the Gibbs free energy of the vapour phase is equal to the Gibbs free energy of the liquid phase for a first order phase transition.

(b) Define the Gibbs free energy in terms of the variables (N, μ) where N is the total number of particles and μ is the chemical potential. What is the value of the chemical potential, μ, for an indeterminable collection of Bose particles (e.g., photons or phonons). Is $\mu > 0$, $\mu < 0$, or $\mu = 0$? Give a clear explanation for the answer that you select.

[II-9] **[2,3,3,2]**

The classical Hamiltonian of the one-dimensional q-state Potts model is given by

$$H = -J\sum_{i=0}^{N-1} \delta_{\sigma_i,\sigma_{i+1}} ,$$

where the $N+1$ "spins" $\sigma_0, \sigma_1, \ldots, \sigma_N$, take the values $\sigma_i = 1,2,\ldots,q$ ($\delta_{m,n}$ is the Kronecker delta). While it may look complicated, you don't have to worry about it, since here, we provide you with the partition function that one can obtain from this Hamiltonian:

$$Z_N(q) = q(e^K + q - 1)^N ,$$

where $K = J/(kT)$.

(a) Define $\hat{Z}_N(q) = Z_N(q)/q^{N+1}$, set $q = N$, and compute

$$\hat{Z}_\infty = \lim_{N\to\infty} \hat{Z}_N(q = N)$$

You will want to utilize the identity $e^x = \lim_{N\to\infty}(1 + x/N)^N$.

(b) Find the free energy F for $\hat{Z}_\infty$.

(c) Find the internal energy U that follows from (b).

(d) Find the heat capacity $C(T)$ that follows from (c), and sketch its behavior.

[II-10] **[3,3,4]**

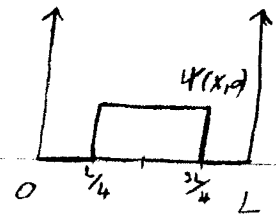
Consider a system consisting of two particles and three energy levels $E_1=0$, $E_2=\varepsilon$ and $E_3=3\varepsilon$.

(a) List all possible arrangements of the system (microsates) if the particles are identical bosons. Calculate the energy of the system for each microstate. Are the energies of the system degenerate?

(b) List all possible arrangements of the system (microstates) if the particles are identical fermions. Calculate the energy of the system for each microstate. Are the energies of the system degenerate?

(c) What is the probability of finding the system at any instant in a doubly occupied state according to Bose-Einstein statistics and according to Fermi-Dirac statistics?

II - 1



$$\hat{H}|\psi_n\rangle = E_n|\psi_n\rangle$$

$$\psi_n(x) = \sqrt{\frac{2}{L}}\sin\left(\frac{n\pi}{L}x\right) \qquad E_n = \frac{\hbar^2}{2m}\left(\frac{n\pi}{L}\right)^2$$

$$|\psi(0)\rangle = \sum_n a_n|\psi_n\rangle$$

$$\psi(x,0) = \begin{cases}\sqrt{\frac{2}{L}} & \frac{L}{4} < x < \frac{3L}{4} \\ 0 & x < \frac{L}{4},\ x > \frac{3L}{4}\end{cases}$$

$$|\psi(t)\rangle = e^{-\frac{i}{\hbar}\hat{H}t}|\psi(0)\rangle = e^{-\frac{i}{\hbar}\hat{H}t}\sum_n a_n|\psi_n\rangle = \sum_n a_n e^{-\frac{i}{\hbar}\hat{H}t}|\psi_n\rangle =$$

$$= \sum_n a_n e^{-\frac{i}{\hbar}E_n t}|\psi_n\rangle$$, where

$$a_n = \langle\psi_n|\psi(0)\rangle = \int_{L/4}^{3L/4}\sqrt{\frac{2}{L}}\sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right)dx = \frac{2}{L}\int_{L/4}^{3L/4}\sin\left(\frac{n\pi x}{L}\right)dx = -\frac{2}{L}\left(\frac{L}{n\pi}\right)\cos\left(\frac{n\pi x}{L}\right)\Bigg|_{L/4}^{3L/4}$$

$$= \frac{2}{n\pi}\left(\cos\left(\frac{n\pi}{4}\right) - \cos\left(\frac{3n\pi}{4}\right)\right) = \begin{cases}\frac{2}{n\pi}\frac{2}{\sqrt{2}}(\pm 1) & n = \text{odd, and } +1: n = 1, 5, 9, \ldots;\ -1: n = 3, 7, 11, \ldots \\ \emptyset & n = \text{even}\end{cases}$$

$$\boxed{\psi(x,t) = \sum_{n=1}^{\infty} a_n e^{-\frac{i}{\hbar}E_n t}\psi_n(x)}$$

$$P_n(t) = \left|a_n e^{-\frac{i}{\hbar}E_n t}\right|^2 = |a_n|^2 = \begin{cases}\frac{8}{n^2\pi^2} & \text{if } n = \text{odd} \\ \emptyset & \text{if } n = \text{even}\end{cases}$$

$$\sum_{n=1}^{\infty} P_n = 1 \quad \text{(can check)}$$

$$P(E_n) = P_n$$

$$P(E > E_1) = P_2 + P_3 + \ldots = 1 - P_1 \overset{!}{=} \boxed{1 - \frac{8}{\pi^2}} \simeq 0.189$$

(1) The total Hamiltonian of the system:

$$H = H_0 + W = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 (1+\rho), \quad \rho << 1$$

Let $\omega'^2 = \omega^2(1+\rho)$

$$E_n = (n+\tfrac{1}{2})\hbar\omega' = (n+\tfrac{1}{2})\hbar\omega\sqrt{1+\rho}$$

Now, if we expand $\sqrt{1+\rho}$,

$$E_n = (n+\tfrac{1}{2})\hbar\omega\left[1+\frac{\rho}{2}-\frac{\rho^2}{8}+\cdots\right]$$

(2) $W = \frac{1}{2}\rho m\omega^2 X^2 = \frac{1}{2}\rho\hbar\omega\hat{X}^2$

Here $\hat{X} \stackrel{def}{=} \frac{1}{\sqrt{2}}(a+a^+)$

and $W = \frac{1}{4}\rho\hbar\omega[a^{+2}+a^2+aa^+ + a^+a]$

By perturbation theory:

$$E_n = E_n^0 + \langle\varphi_n^0|W|\varphi_n^0\rangle + \sum_{n'} \frac{|\langle\varphi_n^0|W|\varphi_{n'}^0\rangle|^2}{E_n^0 - E_{n'}^0}$$

(6)

$$\langle \varphi_n^0 | W | \varphi_n^0 \rangle = \frac{1}{2}\rho \,(n+\tfrac{1}{2})\hbar\omega \qquad (7)$$

$$\langle \varphi_{n+2}^0 | W | \varphi_n^0 \rangle = \frac{1}{4}\rho \sqrt{(n+1)(n+2)}\,\hbar\omega \qquad (8)$$

$$\langle \varphi_{n-2}^0 | W | \varphi_n^0 \rangle = \frac{1}{4}\rho \sqrt{n(n-1)}\,\hbar\omega \qquad (9)$$

We have:

$$E_n = E_n^0 + \frac{\rho}{2}(n+\tfrac{1}{2})\hbar\omega - \frac{\rho^2}{16}(n+1)(n+2)\frac{\hbar\omega}{2} + \frac{\rho^2}{16}n(n-1)\frac{\hbar\omega}{2} + \cdots \qquad (10)$$

#

(a) $\vec{\mu} = -\frac{e}{m}\vec{S} = -\frac{e\hbar}{2m}\vec{\sigma}$

$H = -\vec{\mu}\cdot\vec{B} = \frac{e\hbar}{2m}\vec{B}\cdot\vec{\sigma}$

$= \frac{e\hbar}{2m}\left(B_0\sigma_z + B_1\cos\omega t\,\sigma_x + B_1\sin\omega t\,\sigma_y\right)$

$$\boxed{H = \frac{e\hbar}{2m}\begin{pmatrix} B_0 & B_1 e^{-i\omega t} \\ B_1 e^{i\omega t} & -B_0 \end{pmatrix}}$$

(b)

$$H\Psi = e^{i\lambda t}\begin{pmatrix} B_0 a_1 e^{-i\omega t/2} + B_1 a_2 e^{-i\omega t/2} \\ B_1 a_1 e^{i\omega t/2} - B_0 a_2 e^{i\omega t/2} \end{pmatrix}\frac{e\hbar}{2m}$$

$$i\hbar\partial_t\Psi = e^{i\lambda t}\begin{pmatrix} i(i\lambda - \frac{i\omega}{2})a_1 e^{-i\omega t/2} \\ i(i\lambda + \frac{i\omega}{2})a_2 e^{i\omega t/2} \end{pmatrix}\hbar$$

$$\Rightarrow \left.\begin{array}{l} \frac{\omega_0}{2}a_1 + \frac{\omega_1}{2}a_2 = -a_1(\lambda - \frac{\omega}{2}) \\ \frac{\omega_1}{2}a_1 - \frac{\omega_0}{2}a_2 = -a_2(\lambda + \frac{\omega}{2}) \end{array}\right\} \quad \begin{array}{l} \omega_0 = \frac{eB_0}{m} \\ \omega_1 = \frac{eB_1}{m} \end{array}$$

$$\boxed{\begin{array}{l} a_1\left(\frac{\omega_0}{2} + (\lambda - \frac{\omega}{2})\right) + a_2\frac{\omega_1}{2} = 0 \\ a_1\frac{\omega_1}{2} + a_2\left(-\frac{\omega_0}{2} + (\lambda + \frac{\omega}{2})\right) = 0 \end{array}}$$

(c)

$$\begin{vmatrix} \frac{\omega_0}{2} + (\lambda - \frac{\omega}{2}) & \frac{\omega_1}{2} \\ \frac{\omega_1}{2} & -\frac{\omega_0}{2} + (\lambda + \frac{\omega}{2})) \end{vmatrix} = 0$$

$$= -\frac{\omega_0^2}{4} + (\lambda - \frac{\omega}{2})(\lambda + \frac{\omega}{2}) + \frac{\omega_0}{2}(\lambda + \frac{\omega}{2})$$
$$- \frac{\omega_0}{2}(\lambda - \frac{\omega}{2}) - \frac{\omega_1 2}{4}$$

$$= -\frac{\omega_0 2}{4} - \frac{\omega_0 2}{4} + \lambda^2 - \frac{\omega^2}{4} + \frac{\omega\omega_0}{2}$$

$$\lambda_\pm = \pm\left[\frac{\omega_0 2}{4} + \frac{\omega_1 2}{4} + \frac{\omega^2}{4} - \frac{\omega\omega_0}{2}\right]^{1/2}$$

$$\boxed{\lambda_\pm = \pm\frac{1}{2}\left[(\omega - \omega_0)^2 + \omega_1^2\right]^{1/2}}$$

(d) $\omega = \omega_0 \Rightarrow \lambda_\pm = \pm\frac{1}{2}\omega_1$

Substitute into part (b) eqs:

$$a_{1+}\left(\frac{\omega_0}{2} + \frac{\omega_1}{2} - \frac{\omega_0}{2}\right) + a_{2+}\frac{\omega_1}{2} = 0$$

$$\Rightarrow a_{2+} = -a_{1+}$$

$$a_{1-}\left(\frac{\omega_0}{2} - \frac{\omega_1}{2} - \frac{\omega_0}{2}\right) + a_{2-}\frac{\omega_1}{2} = 0$$

$$\Rightarrow a_{2-} = a_{1-}$$

$$\boxed{\begin{aligned}\Psi_+(t) &= \frac{1}{\sqrt{2}} e^{i\omega_1 t/2}\begin{bmatrix} e^{-i\omega_0 t/2} \\ -e^{i\omega_0 t/2}\end{bmatrix} \\ \Psi_-(t) &= \frac{1}{\sqrt{2}} e^{-i\omega_1 t/2}\begin{bmatrix} e^{-i\omega_0 t/2} \\ e^{i\omega_0 t/2}\end{bmatrix}\end{aligned}}$$

(e) $\Psi_+(0) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad \Psi_-(0) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$\frac{1}{\sqrt{2}}\left(\Psi_+(0) + \Psi_-(0)\right) = \frac{1}{2}\begin{pmatrix} 1+1 \\ 1-1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\Psi(t) = \frac{1}{\sqrt{2}}\left(\Psi_+(t) + \Psi_-(t)\right)$$

$$= \boxed{\begin{bmatrix} \cos\frac{\omega_1 t}{2}\, e^{-i\omega_0 t/2} \\ -i\sin\frac{\omega_1 t}{2}\, e^{i\omega_0 t/2}\end{bmatrix} = \Psi(t)}$$

(f) $\Psi^+(t)\, \frac{\hbar}{2}\sigma_z\, \Psi(t) = \frac{\hbar}{2}\left[\cos^2\frac{\omega_1 t}{2} - \sin^2\frac{\omega_1 t}{2}\right]$

$$= \boxed{\frac{\hbar}{2}\cos\omega_1 t}$$

(II-4) Solution

(a) The scattering potential, or rather ~~the~~ potential energy is equal to $V(r) = B\delta(r)$, where the force center is the origin.

$$B = \int V(r)\, d^3r = \text{constant}.$$

Now the differential cross section for elastic scattering into a unit solid angle (in the center of mass system) is equal to

$$\frac{d\sigma}{d\Omega} = \frac{\mu^2}{4\pi^2\hbar^4}\left|\int V(\vec{r})\, e^{[i(\vec{q}\cdot\vec{r})]}\, d^3\vec{r}\right|^2,$$

where μ is the reduced mass of the colliding particles and $\hbar\vec{q} = \vec{p} - \vec{p}\,'$ is the change in momentum of their relative motion.

From the equations above and the properties of the δ-function we obtain

$$\frac{d\sigma}{d\Omega} = \frac{\mu^2 B^2}{4\pi^2\hbar^4}.$$

The scattering by a delta-function potential is thus isotropic and does not depend on the velocity. It is well known that the same properties characterise the scattering of sufficiently slow particles by a potential well of finite dimensions.

The total scattering cross section is equal to

(II-4) <u>continued</u>.

$$\sigma = 4\pi \frac{d\sigma}{d\Omega} = \frac{\mu^2 B^2}{\pi \hbar^4}$$

where the scattering cross sections are obtained from the general Born equation above, always when

(b) $e^{[i(\vec{q}\cdot\vec{r}_{eff})]} \approx 1$ or $[(\vec{q}\cdot\vec{r}_{eff})] \ll 1$.

In order that this will be the case for all scattering angles, it is necessary (since $q_{max} \approx$ that $k r_{eff} \ll 1$ or $r_{eff} \ll \bar{\lambda}$, where r_{eff} is of the order of the dimensions of the region where $V(r)$ is appreciable different from zero ("range of interaction The delta-function potential considered here is thus an idealized potential with a very small range, or with a range that is much less than the DeBroglie wavelength of the relative motion of the colliding particles. The delta-function potential can describe the interaction of sufficiently slow neutrons with protons or very heavy nuclei.

From the above equations we see that the formal application of the Born approximation leads to the correct result in the case where σ does not depend on the velocity.

Uncertainty: II-5

For a hydrogen atom, the electron potential energy: $V = -\frac{e^2}{4\pi\varepsilon_0 r}$, (1)

The kinetic energy:

$$T \gtrsim T_{min} = \frac{1}{2m_e}(\Delta p)^2 \quad (2)$$

By uncertainty principle: $(\Delta p)^2 \cong \frac{\hbar^2}{r_0^2}$ (3)

$$T_{min} = \frac{1}{2m_e}\frac{\hbar^2}{r_0^2} \quad (4)$$

The lowest ~~total~~ Energy: $E = T_{min} + V = \frac{\hbar^2}{2m_e r_0^2} - \frac{e^2}{4\pi\varepsilon_0 r_0}$

(5)

The minimum energy occurs at

$$r_0 = a_0 = \frac{\hbar^2}{m_e e^2} \cdot 4\pi\varepsilon_0 \quad (6)$$

And is equal to:

$$E_0 = -\frac{m_e e^4}{32\hbar^2}\frac{1}{\varepsilon_0^2\pi^2} \quad (7)$$

II-6

$$\text{probability} = \frac{1}{\hbar^2}\left|\int_0^t V_{Fi}(t')\, e^{i\omega_{Fi}t'}\, dt'\right|^2$$

initial state i : ψ_0

Final state F : ψ_1

$$V_{Fi} = -\alpha \langle \psi_1 | x | \psi_0 \rangle e^{-t^2/\tau^2}$$

$$\text{probability} = \frac{\alpha^2}{\hbar^2}\frac{\hbar}{2m\omega}\pi\tau^2 e^{-\omega^2\tau^2/2}$$

Am

$\boxed{\text{II-7}}$ equation of state:

$$P = -an^2 - \frac{kT}{b}\ln(1-bn) \qquad a, b > 0, \quad n = N/V$$

$$dE = T\,dS - P\,dV \qquad C_V = \left(\frac{\partial E}{\partial T}\right)_V > 0$$

$$\left(\frac{\partial E}{\partial V}\right)_T = T\left(\frac{\partial S}{\partial V}\right)_T - P \qquad \text{Maxwell rel.} \quad \left(\frac{\partial S}{\partial V}\right)_T = \left(\frac{\partial P}{\partial T}\right)_V$$

$$\Downarrow$$

$$\left(\frac{\partial E}{\partial V}\right)_T = T\left(\frac{\partial P}{\partial T}\right)_V - P$$

$E(T,V)$: $$dE = \left(\frac{\partial E}{\partial T}\right)_V dT + \left(\frac{\partial E}{\partial V}\right)_T dV$$

Joule process: E = const., $dE = 0$

$$0 = \left(\frac{\partial E}{\partial T}\right)_V dT + \left(\frac{\partial E}{\partial V}\right)_T dV \Rightarrow \left(\frac{\partial T}{\partial V}\right)_E = -\frac{\left(\frac{\partial E}{\partial V}\right)_T}{\left(\frac{\partial E}{\partial T}\right)_V} = -\frac{\left(\frac{\partial E}{\partial V}\right)_T}{C_V}$$

Thus, for the Joule process:

$$\left(\frac{\partial T}{\partial V}\right)_E = -\frac{T\left(\frac{\partial P}{\partial T}\right)_V - P}{C_V}$$

, and here, C_V = const. > 0

$$\boxed{\Delta T = T_2 - T_1 = -\int_{V_1}^{V_2} \frac{T\left(\frac{\partial P}{\partial T}\right)_V - P}{C_V}\,dV} \qquad V_1 \to V_2 \quad (V_2 > V_1)$$

For the specific system: $\left(\frac{\partial P}{\partial T}\right)_V = -\frac{k}{b}\ln(1-bn)$

$$T_2 - T_1 = -\int_{V_1}^{V_2} \frac{an^2}{C_V}\,dV = -\int_{V_1}^{V_2} \frac{a(N/V)^2\,dV}{C_V} = -\frac{aN^2}{C_V}\int_{V_1}^{V_2}\frac{dV}{V^2} =$$

$$= \frac{aN^2}{C_V}\left(\frac{1}{V_2} - \frac{1}{V_1}\right) \quad (< 0)$$

$\hookrightarrow$ cools down

$$\boxed{T_2 = T_1 + \frac{aN^2}{C_V}\left(\frac{1}{V_2} - \frac{1}{V_1}\right)}$$

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for II-8 a)

TABLE 8.2* Conditions on thermodynamic variables for different systems or processes.

State of System or Type of Process	Valid Equation	Valid Inequality	Equilibrium Condition
S and V constant	$dS = dV = 0$	$dU \leq 0$	Minimum U
S and P constant	$dS = dP = 0$	$dH \leq 0$	Minimum H
T and V constant	$dT = dV = 0$	$dF \leq 0$	Minimum F
T and P constant	$dT = dP = 0$	$dG \leq 0$	Minimum G
Adiabatic	$đQ = dU + P\,dV = 0$	$dS \geq 0$	Maximum S

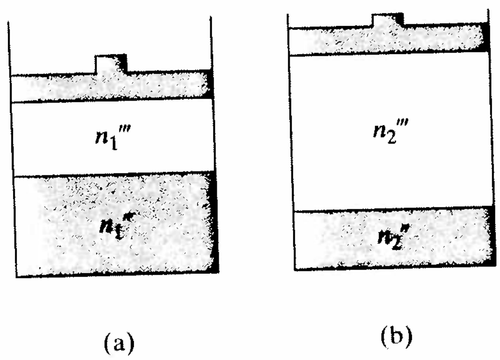


Figure 8.2 Liquid and vapor phases of a substance in equilibrium at temperature T and pressure P; (a) initial state; (b) final state.

phase.† The state of the system is defined in terms of the variables (T, P, n_1'', n_1'''). Consider a second state differing from the first only in the number of kilomoles of liquid and vapor and defined by (T, P, n_2'', n_2''') (Figure 8.2). Mass is conserved so that

$$n_1'' + n_1''' = n_2'' + n_2'''. \tag{8.24}$$

We define g'' and g''' as the specific Gibbs functions of the liquid and vapor, respectively, associated with the particular substance under investigation. Noting that the Gibbs function is an extensive variable, we have for the two states:

$$G_1 = n_1''\,g'' + n_1'''\,g''', \tag{8.25}$$

$$G_2 = n_2''\,g'' + n_2'''\,g'''. \tag{8.26}$$

* Adapted from Table 7.2 in *Thermodynamics and Statistical Mechanics* by P. L. Landsberg, Dover Publications, New York, 1990.

† The notation is that used in section 4.3: one, two, and three primes denote the solid, liquid, and vapor phases, respectively. Here 1 refers to the initial state and 2 to the final state.

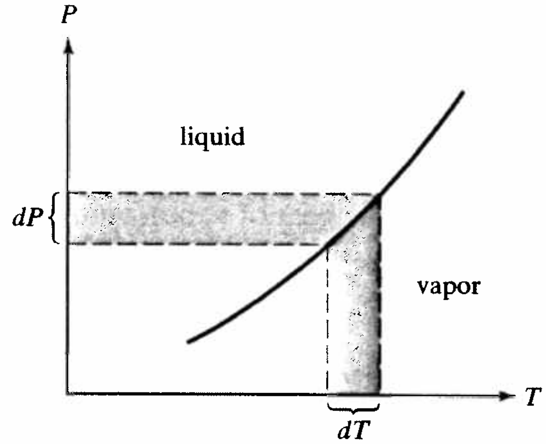


Figure 8.3 Relationship between temperature and pressure for a liquid and vapor in equilibrium. The derivative dP/dT is the slope of the vaporization curve.

Suppose that a reversible transition takes place from state 1 to state 2. Since $(\Delta G)_{T,P} = 0$ for a reversible process, it follows that $G_1 = G_2$. Equating Equations (8.25) and (8.26) and using Equation (8.24), we find that

$$g'' = g'''. \tag{8.27}$$

The specific Gibbs function is the same for the two phases. This is true for all phases in equilibrium, that is, for all points on the curve of the phase transformation (Figure 8.3).

Since at a temperature $T + dT$ and a pressure $P + dP$ we still have equilibrium, it follows that $g'' + dg'' = g''' + dg'''$. Combining this with Equation (8.27), we have

$$dg'' = dg'''.$$

Using the expression for the differential previously derived, we can write

$$-s''\,dT + v''\,dP = -s'''\,dT + v'''\,dP,$$

or

$$(s''' - s'')dT = (v''' - v'')dP.$$

Thus

$$\frac{dP}{dT} = \frac{s''' - s''}{v''' - v''}. \tag{8.28}$$

From the definition of entropy,

$$s''' - s'' = \frac{\ell_{23}}{T}, \tag{8.29}$$

where ℓ_{23} is the latent heat of vaporization. Since heat is absorbed as a liquid becomes a vapor, ℓ_{23} is positive and $s''' > s''$. Substituting Equation (8.29) in Equation (8.28) gives

$$\left(\frac{dP}{dT}\right)_{23} = \frac{\ell_{23}}{T(v''' - v'')} \quad \text{(liquid-vapor).} \tag{8.30}$$

This is the famous Clausius-Clapeyron equation. It gives the *slope* of the curve denoting the boundary between the liquid and vapor phases, that is, the vaporization curve. Similar expressions hold for the sublimation and fusion curves:

$$\left(\frac{dP}{dT}\right)_{13} = \frac{\ell_{13}}{T(v''' - v')} \quad \text{(solid-vapor),} \tag{8.31}$$

$$\left(\frac{dP}{dT}\right)_{12} = \frac{\ell_{12}}{T(v'' - v')} \quad \text{(solid-liquid).} \tag{8.32}$$

The latent heats in these expressions are positive, and the slopes are all positive for substances that expand on melting. A notable exception is water, which contracts when ice melts into liquid; for this case $(dP/dT)_{12} < 0$.

The Clausius-Clapeyron equation, combined with the appropriate equations of state, can in principle yield equations for the phase transformation curves. A simple example is the vaporization curve describing, say, the conversion of liquid water to steam. Here $v''' \gg v''$ (see Chapter 2), and so

$$\left(\frac{dP}{dT}\right)_{23} \approx \frac{\ell_{23}}{Tv'''}.$$

If we treat the vapor as an ideal gas,

$$v''' \approx \frac{RT}{P},$$

so that

$$\frac{dP}{dT} \approx \frac{\ell_{23}\, P}{R\, T^2}.$$

b) Since their number is indeterminate, for fixed T and V, the equilibrium value of N must minimize $F(T,V)$: thus, $\mu = \left(\frac{\partial F}{\partial N}\right)_{T,V} = \underline{\underline{0}}$

Easily shown:

$$Z_N = q\,(e^K + q - 1)^N$$

$$= q^{N+1}\left(1 + \frac{1}{q}(e^K - 1)\right)^N$$

(a)

Then set $q = N$, $\hat{Z}_N = \left(\frac{1}{q^{N+1}} Z_N\right)_{q=N}$,

$$\hat{Z}_\infty = \lim_{N\to\infty}\left(1 + \frac{1}{N}(e^{+K} - 1)\right)^N$$

$$= \exp(e^{+K} - 1).$$

(b) $F = -kT \ln \hat{Z}_\infty = -kT\,(e^{-K} - 1)$

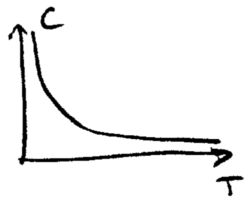
$$= -kT\,(e^{+J/kT} - 1)$$

(c) $U = -T^2 \dfrac{\partial}{\partial T}\dfrac{F}{T}$

$$= -T^2 \frac{\partial}{\partial T}\left(-k(e^{+J/kT} - 1)\right)$$

$$= +T^2 k\left(-\frac{J}{kT^2}\right)e^{J/kT} = -J\,e^{J/kT}$$

(d) $C = \dfrac{dU}{dT} = +\dfrac{J^2}{kT^2}\,e^{J/kT}$



(a) Bosons

0	ε	3ε	E_{micro}	degeneracy
AA	0	0	0	1
0	AA	0	2ε	1
0		AA	6ε	1
A	A	0	ε	1
0	A	A	4ε	1
A	0	A	3ε	1

(b) Fermions

0	ε	3ε	E_{micro}	
A	A	0	ε	1
0	A	A	4ε	1
A	0	A	3ε	1

(c) BE Statistics probability of double occupancy $P = \frac{1}{2}$

FE Statistics probability of double occupancy $P = ($