Physics PhD Qualifying Examination
Part I – Wednesday, January 9, 2008

Name:____
(please print)
Identification Number:________

STUDENT: Designate the problem numbers that you are handing in for grading in the appropriate left hand boxes below. Initial the right hand box.
PROCTOR: Check off the right hand boxes corresponding to the problems received from each student. Initial in the right hand box.

	1	
	2	
	3	
	4	
	5	
	6	
	7	
	8	
	9	
	10	

Student's initials

problems handed in:

Proctor's initials

INSTRUCTIONS FOR SUBMITTING ANSWER SHEETS

1. DO NOT PUT YOUR NAME ON ANY ANSWER SHEET. EXAMS WILL BE COLLATED AND GRADED BY THE ID NUMBER ABOVE.
2. Use at least one separate preprinted answer sheet for each problem. Write on only one side of each answer sheet.
3. Write your **identification number** listed above, in the appropriate box on each preprinted answer sheet.
4. Write the **problem number** in the appropriate box of each preprinted answer sheet. If you use more than one page for an answer, then number the answer sheets with both problem number and page (e.g. Problem 9 – Page 1 of 3).
5. Staple together all the pages pertaining to a given problem. Use a paper clip to group together all eight problems that you are handing in.
6. Hand in a total of *eight* problems. A passing distribution will normally include at least three passed problems from problems 1-5 (Mechanics) and three problems from problems 6-10 (Electricity and Magnetism). **DO NOT HAND IN MORE THAN EIGHT PROBLEMS.**
7. **YOU MUST SHOW ALL YOUR WORK.**

[I-1] **[10]**

Consider a projectile of mass m fired vertically upward in a constant gravitational field g. The direction of the gravitational field is downward. The initial velocity of the projectile is $v(0)=v_0$. Calculate the time t_m required for the projectile to reach maximum height for the case of

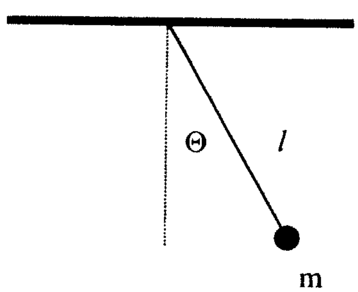
(a) zero resisting force ($F_r = 0$), and

(b) a resisting force proportional to the velocity of the projectile ($F_r = -kmv$).

[I-2] **[10]**

Consider a simple plane pendulum consisting of a mass m attached to a string of length l (the mass of the string is negligible). After the pendulum is set into motion the length of the string is shortened at a constant rate $\frac{dl}{dt} = -\alpha = const.$. The suspension point remains fixed.

(a) Find the Lagrangian L of the pendulum.
(b) Find Lagrange's equation of motion for the pendulum.



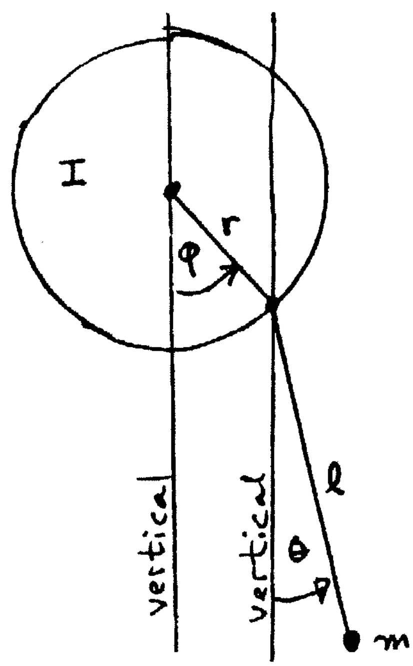
[I-3]   [6,2,2]

Here you will study a disk with moment of inertia I and radius r, rotating on a frictionless pivot about its center of mass. Attached to it by a massless string of length l is a mass m, forming a pendulum that swings under the influence of gravitational acceleration g. See the figure below.

(a) Set up one torque equation for the disk, and two component force equations for the pendulum. Eliminate the tension in the string to obtain two coupled equations of motion. These should be expressed in terms of the two angles ϑ, φ, and their time derivatives, in addition to I, m, r, l, g.

(b) Make a small angles $\vartheta, \varphi << 1$ approximation to linearize the two coupled differential equations. That is, treat ϑ, φ, and their time derivatives as small and keep only the linear order terms in power series expansions.

(c) Determine the two normal mode frequencies of the system in this linear approximation. Express these in terms of the constants of the system I, m, r, l, g.



[I-4]   [10]

Imagine a radial shaft connecting the center of the Earth with its surface. A small object is shot radially outwards from the center of the Earth through this shaft. What is the minimum required initial velocity v_o so that it reaches a distance $2R$ from the center of the Earth?

Assume that the Earth's mass is distributed homogeneously. The Earth's mass is M and its radius is R. The gravitational constant is G. In solving the problem, consider *only* the gravitational effects due to Earth (i.e., neglect the Earth's rotation, friction, air resistance, etc.).

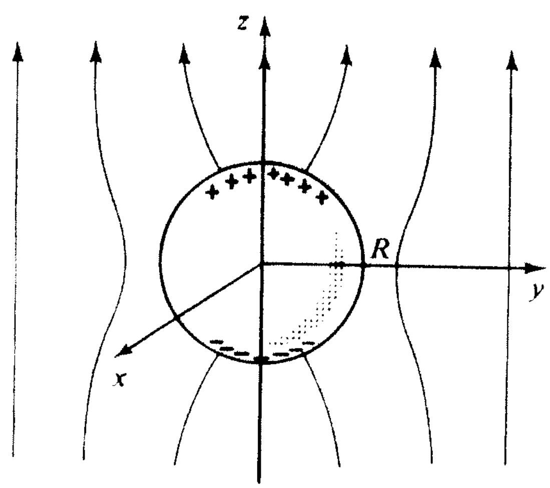
[I-5]   [10]

When cosmic rays enter the earth, muons decay according to the radioactive decay law, $N = N_0 \exp(-0.693t / t_{1/2})$, where N_0 and N are number of muons at time t=0 and t, respectively, and $t_{1/2}$ is the half-life. Let's assume that we mount a detector on top of a 2,000m mountain and count the number of muons traveling at a speed of $v=0.98c$. Over a period of time, we count 1,000 muons. The half-life of muons is known to be $1.52\times10^{-6} s$ in their own rest frame. We move our detector to sea level and measure the number of muons (having $v=0.98c$) detected during an equal period of time. What is the expected number of muons to be measured?

[I-6]   [8,2]

An uncharged metal sphere of radius R is placed in an otherwise uniform electric field $E = E_o \hat{z}$. The field polarizes the sphere, such that it pushes positive charges to the "northern" surface of the sphere, leaving a negative charges on the "southern" surface (see the figure below). This induced charge, in turn, distorts the field in the neighborhood of the sphere.

(a) Find the potential in the region outside the sphere. (Note that this problem has an azimuthal symmetry in the spherical coordinate.)
(b) Also, explain the physical meaning of each term in the solution.



[I-7] **[4,2,2,2]**

(a) Write down all four Maxwell's equations for metallic media. Use Ohm's law to express the current density: $\vec{J} = \sigma \vec{E}$, where σ is the conductivity of the metal.

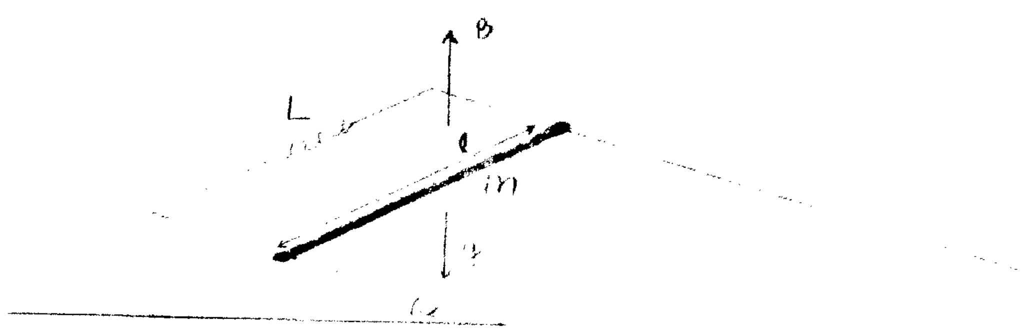
(b) From Maxwell's equations, obtain the modified wave equation for $\vec{E}$. (Hint: Apply the curl to the equation that describes Faraday's law.)

(c) Use the plane wave solution $\vec{E}(x,t) = \vec{E}_0 e^{i(kx-\omega t)}$ to deduce the dispersion relationship of EM wave (i.e. ω versus k). Here, k is the propagating EM wave number.

(d) In a metallic media, k can be a complex number and is given by: $k(\omega,\sigma) = k_1 + ik_2$. Derive the expression for k_1 and k_2.

[I-8] **[10]**

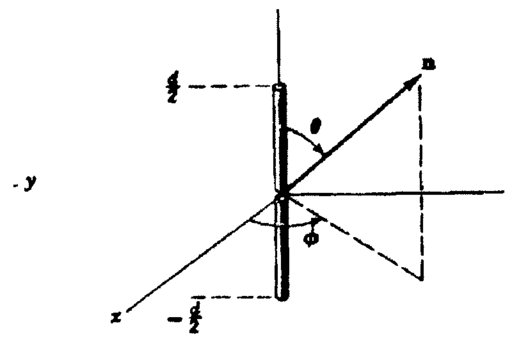
Consider a conducting rod of mass m on a tilted pair of rails in the presence of gravity g and a constant and homogeneous vertical magnetic field B (see sketch below). The angle of inclination is α. The distance between the rails is l. The rails are connected with an inductance L. The resistance of the rod and the rails and the friction between the rod and the rails are negligible. Initially the rod is at rest and there is no current in the loop ($v(0) = 0$, $I(0) = 0$). Describe the motion of the rod, i.e., obtain $v(t)$.



[I-9] **[10]**

Consider a center-fed, linear antenna whose length l is small compared to the radiated wavelength λ. The antenna is assumed to be oriented along the z axis, extending from $z = -d/2$ to $z = d/2$ with a narrow gap at the center for the purpose of excitation. The current is in the same direction in each half of the antenna, having a value I_0 at the gap and falling approximately linearly to zero at the ends:

$$I(z,t) = I_0\left(1 - \frac{2|z|}{d}\right)\exp(-i\omega t)$$



(a) Calculate the linear charge ρ (charge per unit length) density in each arm of the antenna.
(b) Calculate the dipole moment $\boldsymbol{p}$ of the antenna.
(c) Calculate the angular distribution $dP/d\Omega$ of the radiated power.
(d) Calculate the total power P radiated by the antenna.

[I-10] **[6,4]**

(a) Write the relativistic equations of motion for a particle of charge q and mass m in an electromagnetic field. Consider these equations for the special case of motion in the x-direction only, in a Lorentz frame that has a constant electric field E pointing in the positive x-direction.

(b) Show that a particular solution of the equations of motion is given by

$$x = \frac{mc^2}{qE}\cosh\left(\frac{qE\tau}{mc}\right) \quad \text{and} \quad t = \frac{mc}{qE}\sinh\left(\frac{qE\tau}{mc}\right),$$

and show explicitly that the parameter τ used to describe the world-line of the charge q in the above equations is the proper time along this world-line.

Solutions, Part I

I-1

a) *Zero resisting force* ($F_r = 0$):

The equation of motion for the vertical motion is:

$$F = m\,a = m\frac{dv}{dt} = -m\,g \tag{1}$$

Integration of (1) yields

$$v = -gt + v_0 \tag{2}$$

where v_0 is the initial velocity of the projectile and $t = 0$ is the initial time.

The time t_m required for the projectile to reach its maximum height is obtained from (2). Since t_m corresponds to the point of zero velocity,

$$v(t_m) = 0 = v_0 - gt_m\,, \tag{3}$$

we obtain

$$\boxed{t_m = \frac{v_0}{g}} \tag{4}$$

b) *Resisting force proportional to the velocity* $\left(F_r = -km\,v\right)$:

The equation of motion for this case is:

$$F = m\frac{dv}{dt} = -m\,g - km\,v \tag{5}$$

where *-kmv* is a *downward* force for $t < t_m$ and is an *upward* force for $t > t_m$. Integrating, we obtain

$$v(t) = -\frac{g}{k} + \frac{kv_0 + g}{k}\,e^{-kt} \tag{6}$$

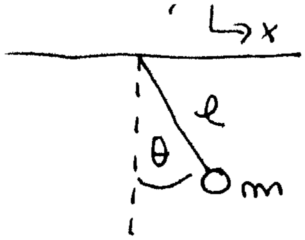
For $t = t_m$, $v(t) = 0$, then from (6),

$$v_0 = \frac{g}{k}\left(e^{kt_m} - 1\right) \tag{7}$$

which can be rewritten as

$$kt_m = \ln\left[1 + \frac{kv_0}{g}\right] \tag{8}$$.

I-2



1 object, 2-dim. motion,
2 coordinates, 1 degree of freedom
x, y ↳ one generalized coordinate θ

$$l(t) = l_0 - \alpha t$$

$$\frac{dl}{dt} = -\alpha$$

polar coordinates $\quad x = l\sin\theta \;,\; y = -l\cos\theta$

$$T = \frac{1}{2}m(\dot{x}^2+\dot{y}^2)$$

$\cancel{\dot{x} = \dot{l}\cos\theta}$

$$\dot{x} = \dot{l}\sin\theta + l\dot{\theta}\cos\theta = -\alpha\sin\theta + l\dot{\theta}\cos$$

$$\dot{y} = -\dot{l}\cos\theta + l\dot{\theta}\sin\theta = -\dot{l}\cos\theta + l\dot{\theta}\sin$$

$$T = \frac{1}{2}m\left(\dot{l}^2 + l^2\dot{\theta}^2\right) = \frac{1}{2}m\left(\alpha^2 + (l_0-\alpha t)^2\dot{\theta}^2\right)$$

$$U = (l - l\cos\theta)mg = (l_0-\alpha t)mg(1-\cos\theta)$$

$$L = T - U$$

$$L = \frac{1}{2}m\left(\alpha^2 + (l_0-\alpha t)^2\dot{\theta}^2\right) - mg(l_0-\alpha t)(1-\cos\theta)$$

$$\frac{\partial L}{\partial\dot{\theta}} = m(l_0-\alpha t)^2\dot{\theta}$$

$$\frac{d}{dt}\frac{\partial L}{\partial\dot{\theta}} = m(l_0-\alpha t)^2\ddot{\theta} + 2m(l_0-\alpha t)(-\alpha)\dot{\theta}$$

$$\frac{\partial L}{\partial\theta} = -mg(l_0-\alpha t)\sin\theta$$

$$\cancel{m}(l_0-\alpha t)^2\ddot{\theta} - 2\cancel{m}\alpha(l_0-\alpha t)\dot{\theta} + \cancel{m}g(l_0-\alpha t)\sin\theta = 0$$

$$\ddot{\theta} - \frac{2\alpha}{(l_0-\alpha t)}\dot{\theta} + \frac{g}{(l_0-\alpha t)}\sin\theta = 0$$

I-2

$$L = \frac{1}{2} m (\dot{l}^2 + l^2 \dot{\theta}^2) - mgl(1-\cos\theta)$$

$$\frac{\partial L}{\partial \dot{l}} = m\dot{l} \qquad \frac{d}{dt}\frac{\partial L}{\partial \dot{l}} = m\ddot{l} \qquad \frac{\partial L}{\partial l} = -mg(1-\cos\theta) + ml\dot{\theta}^2$$

$$m\ddot{l} + mg(1-\cos\theta) - ml\dot{\theta}^2 = 0$$

$$\frac{\partial L}{\partial \dot{\theta}} = ml^2\dot{\theta} \qquad \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = ml^2\ddot{\theta} + 2ml\dot{l}\dot{\theta}$$

$$\frac{\partial L}{\partial \theta} = -mg\sin\theta$$

$$ml^2\ddot{\theta} + 2ml\dot{l} + mg\sin\theta = 0$$

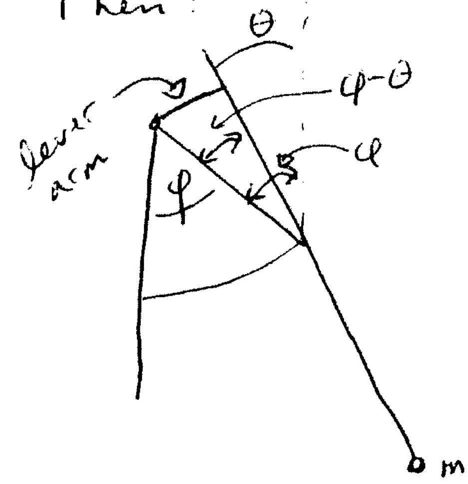
[redacted]

[redacted]

[redacted]

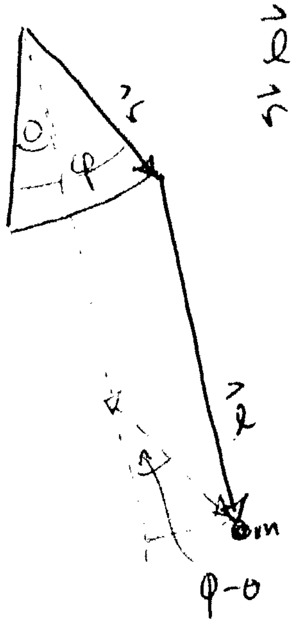
(I-3) Soln.

(a) Let T = tension in string. Then:



$$\boxed{I\ddot{\varphi} + Tr\cdot\sin(\varphi-\theta) = 0.} \quad (*)$$

For pendulum



$$\vec{l} = \hat{r}\,l$$

$$\vec{r} = r\cos(\varphi-\theta)\,\hat{r} + r\sin(\varphi-\theta)\,\hat{\theta}$$

$$\dot{\hat{r}} = \hat{\theta}\,\ddot{\theta}, \quad \dot{\hat{\theta}} = -\hat{r}\,\ddot{\theta}$$

$$\vec{R} = \vec{l} + \vec{r}$$

$$\ddot{\vec{R}} = \hat{r}\left(-l\dot{\theta}^2 - r\dot{\varphi}^2\cos(\varphi-\theta) - r\ddot{\varphi}\sin(\varphi-\theta)\right) + \hat{\theta}\left(l\ddot{\theta} + r\ddot{\varphi}\cos(\varphi-\theta) - r\dot{\varphi}^2\sin(\varphi-\theta)\right)$$

$$\boxed{\begin{aligned} m\left[-l\dot{\theta}^2 - r\dot{\varphi}^2\cos(\varphi-\theta) - r\ddot{\varphi}\sin(\varphi-\theta)\right] &= mg\cos\theta - T \\ m\left[l\ddot{\theta} + r\ddot{\varphi}\cos(\varphi-\theta) - r\dot{\varphi}^2\sin(\varphi-\theta)\right] &= -mg\sin\theta \end{aligned}} \quad (**)$$

Eliminate T between (*) & (**):

$$\boxed{\begin{aligned} 0 &= I\ddot{\varphi} + r\sin(\varphi-\theta)\left\{mg\cos\theta + m\left[l\dot{\theta}^2 + r\dot{\varphi}^2\cos(\varphi-\theta) + r\ddot{\varphi}\sin(\varphi-\theta)\right]\right\} \\ 0 &= m\left[l\ddot{\theta} + r\ddot{\varphi}\cos(\varphi-\theta) - r\dot{\varphi}^2\sin(\varphi-\theta)\right] + mg\sin\theta \end{aligned}}$$

(b)

$$\boxed{\begin{aligned} 0 &\approx I\ddot{\varphi} + mgr(\varphi - \theta) \\ 0 &\approx l\ddot{\theta} + r\ddot{\varphi} + g\theta \end{aligned}}$$

(c)

$$\left.\begin{aligned} 0 &\approx -\omega^2 I A_\varphi + mgr(A_\varphi - A_\theta) \\ 0 &\approx -\omega^2 l A_\theta - \omega^2 r A_\varphi + g A_\theta \end{aligned}\right\}$$

$$\begin{pmatrix} mgr - \omega^2 I & -mgr \\ -\omega^2 r & -\omega^2 l + g \end{pmatrix} \begin{pmatrix} A_\varphi \\ A_\theta \end{pmatrix}$$

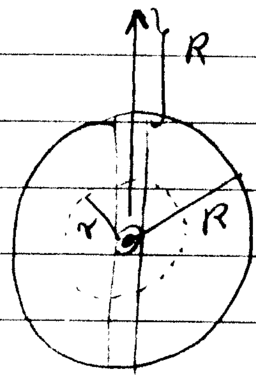
$$0 = \begin{vmatrix} mgr - \omega^2 I & -mgr \\ -\omega^2 r & -\omega^2 l + g \end{vmatrix}$$

$$= Il\,\omega^4 - (mgrl + mgr^2 + gI)\omega^2 + mg^2 r$$

$$\boxed{\omega_\pm^2 = \left\{(mgrl + gI + mgr^2) \pm \left[(mgrl + gI + mgr^2)^2 - 4Ilmg^2 r\right]^{1/2}\right\} \Big/ \left\{2Il\right\}}$$

$\boxed{\text{I-4}}$

$\rho = \dfrac{M}{\frac{4\pi}{3}R^3}$



Gauss' law:

$r < R$ inside Earth: $\oint \vec{g} \cdot d\vec{A} = -4\pi G M_{encl}$

$$g(r)\, 4\pi r^2 = -4\pi G \rho \frac{4\pi}{3} r^3$$

$$g(r) = -G\rho \frac{4\pi}{3} r = -G\frac{M}{R^3} r \qquad r <$$

$r < R$: $\quad \phi_<(r) = G\dfrac{M}{R^3}\dfrac{r^2}{2} + C$

$r > R$: $\quad \phi_>(r) = -G\dfrac{M}{r} \qquad$ (e.g., with $\phi(\infty) = 0$)

Potential must be a continuous function: $r = R$: $\phi_<(R) = \phi_>(R)$

$$G\frac{M}{R^3}\frac{R^2}{2} + C = -G\frac{M}{R}$$

$$\frac{1}{2}G\frac{M}{R} + C = -G\frac{M}{R} \implies C = -\frac{3}{2}G\frac{M}{R}$$

$$\boxed{\phi(r) = \begin{cases} \frac{1}{2}G\frac{M}{R^3}r^2 - \frac{3}{2}G\frac{M}{R} & r < R \\ -G\frac{M}{r} & r > R \end{cases}}$$

- initial condition: $r = 0$, $v = v_0$
- to reach final position $r = 2R$ $(v = 0)$

$$\frac{1}{2}mv_0^2 - \frac{3}{2}G\frac{M}{R}m = 0 - G\frac{M}{2R}\cancel{m}$$

$$v_0^2/2 = \frac{GM}{R} \implies \boxed{v_0 = \sqrt{\frac{2GM}{R}}}$$

(I-5)

(1) In the rest fram, the radioactive decay law: $N = N_0 e^{-0.693 \Delta t / t_{1/2}}$ .... ①

(2) Given that $N_0 = 1{,}000$, $t_{1/2} = 1.52 \times 10^{-6} s$ .... ②

(3) Also, given that an observer on earth observes the time-of-flight, $\Delta t'$,

$$\Delta t' = \frac{2000m - 0m}{v} = \frac{2000m}{0.98c} .$$

$$\Delta t' = 6.8 \times 10^{-6} s \quad .... ③$$

(4) From Lorentz Transformation,

$$\Delta t' = \gamma \Delta t$$

where $\gamma \overset{def}{=} \frac{1}{\sqrt{1 - v^2/c^2}} = 5.02$

$\therefore \Delta t' = 5 \Delta t$ and $\Delta t = 1.35 \times 10^{-6} s$ ... ④

(5) Combine equ. ①, ② and ④,

We have $N = 539$ #

(I-6)

(1) In spherical coordinate, the general solution to the Laplace equation (with azimuthal symmetry) is a linear combination of:

$$V(r,\theta) = \sum_{l=0}^{\infty} \left(A_l r^l + \frac{B_l}{r^{l+1}}\right) P_l(\cos\theta) \cdots\cdots (1$$

where $P_l(\cos\theta)$ is the Legendre polynomials.

(2) Since the metal sphere is an equipotential and at large z (i.e. far field) the field is E_0:

The boundary conditions for the problem are:

$$\begin{cases} V = 0 & \text{when } r = R \quad \cdots\cdots (2) \\ V \to -E_0 r \cos\theta & \text{for } r \gg R \quad \cdots\cdots (3) \end{cases}$$

(3) From equ. (1) and (2) $\Rightarrow$ $A_l R^l + \frac{B_l}{R^{l+1}} = 0$

$$\therefore B_l = -A_l R^{2l+1}$$

We have $V(r,\theta) = \sum_{l=0}^{\infty} A_l \left(r^l - \frac{R^{2l+1}}{r^{l+1}}\right) P_l(\cos\theta) \cdots (4)$

(4) For $r \gg R$, the second term in parentheses is negligible.

From (3) and (4) $\Rightarrow \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta) = -E_0 r \cos\theta \cdots (5)$

(1-b)

Since $P_{l=1}(\cos\theta) = \cos\theta$, $A_1 = -E_0$ ...... (6)
all other A_l's zero

(5) Combining equ. (4) and (6),

we have: $V(r,\theta) = -E_0\left(r - \frac{R^3}{r^2}\right)\cos\theta$ .... (7)

(6) The 1st term $(-E_0 r\cos\theta)$ is due to the external field.

The 2nd term $\left(E_0\frac{R^3}{r^2}\cos\theta\right)$ is due to the induced charge on the metal surface.

#

(I-17)

(a) Maxwell equ. in a metal, where $\vec{J} = \sigma \vec{E}$, are:

(i) $\nabla \cdot \vec{E} = 0$

(ii) $\nabla \cdot \vec{B} = 0$

(iii) $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$

(iv) $\nabla \times \vec{B} = \mu \epsilon \frac{\partial \vec{E}}{\partial t} + \mu(\sigma \vec{E})$.

(b) Apply the curl to (iii),

we have: $\nabla^2 \vec{E} = \mu \epsilon \frac{\partial^2 \vec{E}}{\partial t^2} + \mu(\sigma \frac{\partial \vec{E}}{\partial t})$. .... (V)

(c) $\vec{E}(x,t) = \vec{E}_0 e^{i(kx - \omega t)}$,

From

Equation-(V) becomes: $k^2 = \mu \epsilon \omega^2 + i(\mu \sigma \omega)$; ...(VI)

(d) From $k = k_1 + i k_2$,

$\Rightarrow k^2 = (k_1^2 - k_2^2) + 2 i k_1 k_2$ .... (VII)

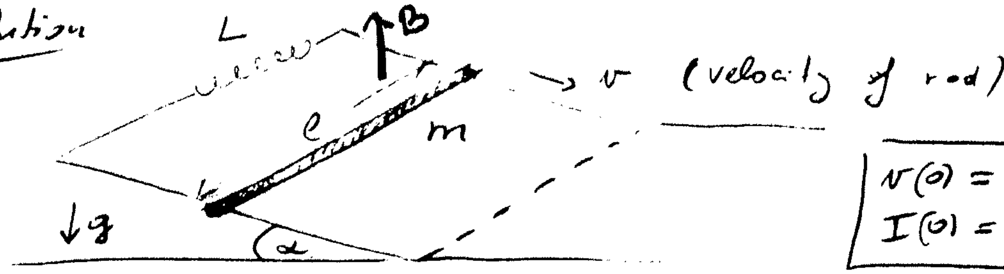
Solve k_1 and k_2 from equ. (VI) and (VII)

$\therefore k_1 = \omega \sqrt{\frac{\epsilon \mu}{2}} \left(\sqrt{1 + \left(\frac{\sigma}{\epsilon \omega}\right)^2} + 1 \right)^{1/2}$

$\therefore k_2 = \omega \sqrt{\frac{\epsilon \mu}{2}} \left(\sqrt{1 + \left(\frac{\sigma}{\epsilon \omega}\right)^2} - 1 \right)^{1/2}$

#

I-8 / Solution



$$\boxed{\begin{matrix} v(0) = 0 \\ I(0) = 0 \end{matrix}}$$ initial

(1) $m\dot{v}(t) = mg\sin\alpha - I(t)\ell B\cos\alpha$ (Newton's II.)

(2) $v(t)\ell B\cos\alpha - L\dot{I}(t) = 0$ (voltages in closed loop

$$m\ddot{v} = -\dot{I}\ell B\cos\alpha = -\frac{v\ell B\cos\alpha}{L}\cdot \ell B\cos\alpha$$

$$m\ddot{v} = -\frac{\ell^2 B^2\cos^2\alpha}{L}v(t)$$

$$\ddot{v} = -\frac{\ell^2 B^2\cos^2\alpha}{Lm}v(t) \Rightarrow \boxed{\omega = \frac{\ell B\cos\alpha}{\sqrt{mL}}}$$

(3) $\begin{cases} v(t) = A\cos(\omega t) + B\sin(\omega t) \\ \dot{v}(t) = -\omega A\sin(\omega t) + \omega B\cos(\omega t) \end{cases}$ (harmonic oscillati

need $v(0)$ and $\dot{v}(0)$

$v(0) = 0$

and from $I(0) = 0$ & (1): $m\dot{v}(0) = mg\sin\alpha \Rightarrow \dot{v}(0) =$

$t=0$: from (3): $0 = A$

$g\sin\alpha = \omega B \Rightarrow B = \frac{g\sin\alpha}{\omega} = \frac{\sqrt{mL}\cdot g\sin\alpha}{\ell B\cos\alpha}$

$$\boxed{v(t) = \frac{g\sin\alpha\sqrt{mL}}{\ell B\cos\alpha}\cdot\sin\left(\frac{\ell B\cos\alpha}{\sqrt{mL}}t\right)}$$

I-9

$$I(z,t) = I_0\left(1 - \frac{2|z|}{d}\right)e^{-i\omega t}$$

linear charge density ϱ in each arm of the antenna from continuity equation

$$i\omega\varrho = \vec{\nabla}\cdot\vec{j} = \frac{\partial I}{\partial z} = \pm\frac{2iI_0}{\omega d}$$

dipole moment $\vec{p} = p\,\vec{e}_z$

$$p = \int_{-d/2}^{d/2} z\,\varrho(z)\,dz = \frac{iI_0 d}{2\omega}$$

angular distribution of radiated power

$$\frac{dP}{d\Omega} = \frac{c^2 Z_0}{32\pi^2}k^4\,|\vec{p}|^2\sin^2\theta$$

$$Z_0 = \sqrt{\frac{\varepsilon_0}{\mu_0}} \qquad k = \frac{2\pi}{\lambda}$$

$$\frac{dP}{d\Omega} = \frac{Z_0 I_0^2}{128\pi^2}(kd)^2\sin^2\theta$$

total power radiated

$$P = \frac{Z_0 I_0^2 (kd)^2}{48\pi}$$

compare also Jackson, Classical Electrodynamics 3rd ed. p. 412

Jan 2008.

I-10. Solution:

a) Starting from a 4-vector potential $(\phi, \vec{A})$, we obtain the equation of motion for a charged particle in the electromagnetic field

$$\frac{d\vec{p}}{dt} = -\frac{q}{c}\frac{\partial \vec{A}}{\partial t} - q\vec{\nabla}\phi - \frac{q}{c}\vec{v}\times(\vec{\nabla}\times\vec{A})$$

by definition: $$\vec{E} = -\frac{1}{c}\frac{\partial \vec{A}}{\partial t} - \vec{\nabla}\phi$$
$$\vec{H} = \vec{\nabla}\times\vec{A}$$

hence, $$\frac{d\vec{p}}{dt} = q\vec{E} + \frac{q}{c}\vec{v}\times\vec{H}$$

In this case of one-dimensional motion where there is only an electric field E and momentum p in the x-direction we obtain

$$\frac{dp}{dt} = qE \qquad \text{where}$$

$$p = \frac{mv}{\sqrt{1-\beta^2}} = \frac{mc\beta}{\sqrt{1-\beta^2}}, \quad \beta = v/c.$$

b) To show that the given equations are solutions of above equation for (dp/dt), we write $$\beta c = \frac{dx}{dt} = \frac{dx/d\tau}{dt/d\tau}$$

[I-10] solution-continued

now $\dfrac{dx}{d\tau} = c \sinh(qE\tau/mc)$

$$\frac{dt}{d\tau} = \cosh(qE\tau/mc) \quad \text{and}$$

$\beta c = c \tanh(qE\tau/mc)$ using $(1-\tanh^2 = 1/\cosh^2$

$$p = \frac{mc \tan(qE\tau/mc)}{1/\cosh(qE\tau/mc)} = mc \sinh(qE\tau/mc)$$

and

$$\frac{dp}{dt} = \frac{\frac{dp}{d\tau}}{\frac{dt}{d\tau}} = \frac{qE\cosh(qE\tau/mc)}{\cosh(qE\tau/mc)} = qE$$

which varifies $dp/dt = qE$ in part(a). Now to show that "τ" is the proper time for the particle, we must demonstrate that

$c^2 d\tau^2 = c^2 dt^2 - dx^2$; using the equations given in the problem for x and t,

$\dfrac{dx}{} = c \sinh(qE\tau/mc) d\tau$, $dt = \cosh(qE\tau/mc) d\tau$

so

$$c^2 dt^2 - dx^2 = c^2(\cosh^2 - \sinh^2) d\tau = c^2 d\tau$$

as required.

Physics PhD Qualifying Examination
Part II – Friday, January 11, 2008

Name:____
(please print)
Identification Number:________

STUDENT: insert a check mark in the left boxes to designate the problem numbers that you are handing in for grading.
PROCTOR: check off the right hand boxes corresponding to the problems received from each student. Initial in the right hand box.

	1	
	2	
	3	
	4	
	5	
	6	
	7	
	8	
	9	
	10	

Student's initials
problems handed in:
Proctor's initials

INSTRUCTIONS FOR SUBMITTING ANSWER SHEETS

1. DO NOT PUT YOUR NAME ON ANY ANSWER SHEET. EXAMS WILL BE COLLATED AND GRADED BY THE ID NUMBER ABOVE.
2. Use at least one separate preprinted answer sheet for each problem. Write on only one side of each answer sheet.
3. Write your **identification number** listed above, in the appropriate box on the preprinted sheets.
4. Write the **problem number** in the appropriate box of each preprinted answer sheet. If you use more than one page for an answer, then number the answer sheets with both problem number and page (e.g. Problem 9 – Page 1 of 3).
5. Staple together all the pages pertaining to a given problem. Use a paper clip to group together all eight problems that you are handing in.
6. Hand in a total of *eight* problems. A passing distribution will normally include at least four passed problems from problems 1-6 (Quantum Physics) and two problems from problems 7-10 (Thermodynamics and Statistical Mechanics). **DO NOT HAND IN MORE THAN EIGHT PROBLEMS.**
7. **YOU MUST SHOW ALL YOUR WORK.**

[II-1]   [5,3,2]

The eigenfunctions for a potential of the form

$$V(x) = \begin{cases} \infty & x < 0 \\ 0 & 0 < x < a \\ \infty & x > a \end{cases}$$

are given by

$$U_n(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right) .$$

Suppose a particle in this potential has an initial normalized wave function of the form

$$\psi(x,0) = A\left[\sin\left(\frac{\pi x}{a}\right)\right]^3 .$$

(a) What is the form of $\psi(x,t)$?
(Hint-1: expand the initial wavefunction into a linear combination of $U_n(x)$.)
(Hint-2: You may use the expression $\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$.)

(b) Calculate A without doing the integral $\int d\theta \sin^6\theta$ [i.e., by using $\psi(x,0)$ expressed as a linear combination of the eigenfunctions $U_n(x)$].

(c) What is the probability that an energy measurement yields E_3, where $E_n = \frac{n^2\pi^2\hbar^2}{2ma^2}$.

[II-2] **[4,2,4]**

A hydrogen atom is placed in a uniform external electric $\boldsymbol{E}_o = E_o\hat{z}$.

(a) Compute the first-order shift in energy levels for the $n = 2$ states. Here and in all other parts below, express your answers in terms of the Bohr radius a, the electronic charge e and the external field strength E_o. Utilize the following unperturbed wavefunction forms:

$$R_{20} = \frac{1}{\sqrt{2}} a^{-3/2}\left(1 - \frac{r}{2a}\right)e^{-r/2a}\ ,\quad R_{21} = \frac{1}{\sqrt{24}} a^{-3/2}\frac{r}{a}e^{-r/2a}\ ,$$

$$Y_{00} = \frac{1}{\sqrt{4\pi}}\ ,\quad Y_{10} = \sqrt{\frac{3}{4\pi}}\cos(\theta)\ ,\quad Y_{1,\pm1} = \mp\sqrt{\frac{3}{8\pi}}\sin(\theta)e^{\pm i\varphi}\ .$$

(b) Give the first-order energy eigenstates that result from the mixing in part (a).

(c) Suppose the system is prepared initially in the ψ_{200} state ($n = 2,\ l = 0,\ m = 0$) and an electric field is switched on instantaneously at $t = 0$:

$$\boldsymbol{E}(t) = \begin{cases} 0 & t < 0 \\ E_o\hat{z} & t \geq 0 \end{cases}\ .$$

Using the results of parts (a) and (b), compute the probability as a function of time for the atom to be observed with orbital angular momentum $\boldsymbol{L} = 0$.

[II-3] **[3,4,3]**

Two spin-1/2 particles are separated by a distance vector $\boldsymbol{a} = a\hat{z}$ ($a = |\boldsymbol{a}|$) and interact through the magnetic dipole energy

$$H = \frac{(\boldsymbol{\mu}_1 \cdot \boldsymbol{\mu}_2)}{a^3} - 3\frac{(\boldsymbol{\mu}_1 \cdot \boldsymbol{a})(\boldsymbol{\mu}_2 \cdot \boldsymbol{a})}{a^5}\ ,$$

where $\boldsymbol{\mu}_j$ is the magnetic moment of spin j. The system of two spins consists of eigenstates of total spin S^2 and total S_z.

(a) Write the Hamiltonian in terms of the spin operators.
(b) Write the Hamiltonian in terms of S^2 and S_z.
(c) Give the eigenvalues for all states.

[II-4] **[3,4,3]**

In this problem you will model elastic scattering from a cubic lattice of scatterers. The potential energy is:

$$V(\boldsymbol{x}) = \sum_{n_1=-M}^{M} \sum_{n_2=-M}^{M} \sum_{n_3=-N}^{N} \delta(\boldsymbol{x} - a\boldsymbol{n}) \quad ,$$

where $\boldsymbol{n} = (n_1, n_2, n_3)$. Here a is the lattice spacing, $a\boldsymbol{n}$ describe the locations of the lattice sites; there are $2M+1$ lattice sites in the both the $\hat{x}$ and $\hat{y}$ directions (corresponding to n_1 and n_2 respectively), and there are $2N+1$ sites in the $\hat{z}$ direction, corresponding to n_3. The incoming plane wave has wave vector $\boldsymbol{k} = k\hat{\boldsymbol{z}}$.

(a) Determine the scattering amplitude $f(\boldsymbol{k}', \boldsymbol{k})$ in the (first order) Born approximation, written as a sum over the lattice sites $\boldsymbol{n}$. Here $\boldsymbol{k}'$ is the wave vector of the scattered particle.

(b) Now perform the sum exactly. Use the geometric progression:

$$\sum_{j=1}^{N} z^{j-1} = \frac{z^N - 1}{z - 1}, \quad z \neq 1.$$

(c) Let θ be the scattering angle. The $\hat{z}$ component of the vector $\boldsymbol{q} = \boldsymbol{k} - \boldsymbol{k}'$ is $q_3 = 2k\sin^2(\theta/2)$. Using the result of part (b), show that there are special values of q_3 for which the scattering amplitude $f(\boldsymbol{k}', \boldsymbol{k})$ vanishes. Determine the corresponding values of θ for which $f(\boldsymbol{k}', \boldsymbol{k})$ vanishes.

[II-5] **[10]**

Estimate the ground state energy of a two electron atom with nuclear charge Z, using the Heisenberg Uncertainty Relations. Evaluate the ground state energy for Li^+ from your calculations (Li^+ has 3 protons and 2 electrons).
The ground state energy of an ion with two electrons and nuclear charge Z should be expressed as a function of Z and include the Rydberg constant. The Rydberg constant Ry is given by $Ry = \frac{\mu e^4}{2\hbar^2} = 13.6\mathrm{eV}$, where μ is the reduced mass, e is the electron charge, and $\hbar$ is Planck's constant.

[II-6] **[10]**

$\Psi(x, t)$ is a solution of the time-dependent Schroedinger equation

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\psi + V(\vec{x})\psi .$$

Show that $\rho(\vec{x},t) = |\psi(x,t)|^2$ and $\vec{j}(\vec{x},t) = \frac{i\hbar}{2m}\left(\psi\vec{\nabla}\psi^* - \psi^*\vec{\nabla}\psi\right)$ satisfy the continuity equation

$$\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot\vec{j} = 0 .$$

[II-7] **[10]**

When a particular one-component material is in phase I, it obeys the equation of state

$$\beta p = a + b\beta\mu \ ,$$

where $\beta = 1/T$, p and μ are the pressure and chemical potential, respectively, and a and b are positive functions of β. When the material is in phase II,

$$\beta p = c + d(\beta\mu)^2,$$

where c and d are positive functions of β, $d > b$, and $c < a$. Determine the density change that occurs when the material undergoes a phase transformation from phase I to phase II. What is the pressure at which the transition occurs?

[II-8] **[2,4,4]**

The Helmholtz free energy of a dilute plasma gas consisting of N electrons confined to a volume V at temperature T is given by

$$F(T,V,N) = F^{ideal}(T,V,N) - \frac{2}{3}Ne^2\left(\frac{4\pi Ne^2}{Vk_BT}\right)^{1/2},$$

where e is the electron charge, k_B is Boltzmann's constant, and $F^{ideal}(T,V,N)$ is the Helmholtz free energy of a mono-atomic ideal gas.

(a) Obtain the equation of state of the above plasma gas.
(b) Obtain the internal energy of the system $E(T,V,N)$.
(c) Obtain the constant-volume heat capacity C_V.

Note: your answers should be fully explicit in terms of the variables T, V, and N. To that end, you are expected to remember and use the equation of state, internal energy, heat capacity of the ideal gas in obtaining your final results.

[II-9] **[4,4,2]**

A cubic volume $V = L^3$ of an ideal monoatomic gas is at equilibrium in a uniform gravitational field $\mathbf{g} = -g\hat{z}$. Each atom has mass m. Take the bottom of the container to be $z = 0$ and the top to be $z = L$.

(a) Determine the classical partition function for the gas.
(b) Determine the equation of state.
(c) Show that this reduces to the usual expression in the $g \to 0$ limit.

[II-10] **[3,7]**

Consider a model system with single-particle energy levels $\varepsilon, 2\varepsilon, 3\varepsilon, 4\varepsilon, \ldots$. The system is completely isolated from the rest of the universe; there are $N = 3$ electrons in the system, and the total energy of the system is $E = 6\varepsilon$. The only degeneracy of the energy levels is associated with the spins of the particles.

(a) What is the entropy of the system subject to the above constraints?
(b) What is the expectation value of the number of electrons in each of the single-particle energy levels?

Solutions, Part II

(II-1)

(a)

Given: $\psi(x,t) = \sum_n \alpha_n U_n(x)\, e^{-iE_n t/\hbar}$ ... ①

$\psi(x,0) = \sum_n \alpha_n U_n(x)$ ... ②

Let's expand $\psi(x,0)$ and represent it using the $\{U_n(x)\}$ - eigenfunction.

note: $\sin\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2i}$

$\sin^3\theta = -\frac{1}{4}\sin 3\theta + \frac{3}{4}\sin\theta$ ----- ③

We have: $\psi(x,0) = A\left[\frac{3}{4}\sin\frac{\pi x}{a} - \frac{1}{4}\sin\frac{3\pi x}{a}\right]$

$\therefore\ \psi(x,0) = \left(\frac{3}{4}A\sqrt{\frac{a}{2}}\right) U_1(x) - \left(\frac{1}{4}A\sqrt{\frac{a}{2}}\right) U_3(x)$

.... ④

Combining equ. ① and ④,

$$\psi(x,t) = \left(\frac{3}{4}A\sqrt{\frac{a}{2}}\right) U_1(x)\, e^{-iE_1 t/\hbar} - \left(\frac{1}{4}A\sqrt{\frac{a}{2}}\right) U_3(x)\, e^{-iE_3 t/\hbar}$$

.... ⑤

where $E_1 = \dfrac{\pi^2\hbar^2}{2ma^2}$, $E_3 = \dfrac{9\pi^2\hbar^2}{2ma^2}$ .... ⑥

#

(II-1)

(b)

To compute the normalization constant, A,

$$\psi(x,0) = \frac{3A}{4}\sqrt{\frac{a}{2}}\,U_1(x) - \frac{1}{4}A\sqrt{\frac{a}{2}}\,U_3(x)$$

$$\psi(x,0) = \quad C_1 U_1(x) + C_3 U_3(x)$$

Demand $|C_1|^2 + |C_3|^2 = 1$ .... (7)

We have $A = \sqrt{\frac{16}{5a}}$ .... (8)

#

(c) The probability that energy measurement yields E_3 is: $|C_3|^2 = \frac{A^2}{32} \cdot a = \frac{1}{10}$ .... (9)

#

(a) (II-2) Soln

$\Delta V = -e\mathcal{E}_0 z = -e\mathcal{E}_0 r\cos\theta$

Need $Y^*_{\ell m} Y_{\ell' m'}$ to be odd under $\cos\theta \to -\cos\theta$ in order to get a non-vanishing result in

$$\int_{-1}^{1} d(\cos\theta)\,\cos\theta\, Y^*_{\ell m} Y_{\ell' m'} \,.$$

$\Rightarrow$ Y^*_{10} with either Y_{00} or $Y_{1,\pm1}$ (& c.c.)

But $\int_0^{2\pi} d\varphi$ will vanish for $Y_{1,\pm1}$. This leaves only:

$$2\pi\int_{-1}^{1} Y^*_{00} Y_{10}\cos\theta\, d(\cos\theta) = \frac{1}{\sqrt{3}}$$

The radial integral is:

$$\int_0^\infty r^2 dr\, R_{20}R_{21}\, r = \frac{1}{\sqrt{48}}\, a^{-3}\int_0^\infty r^3 dr\left(\frac{r}{a} - \frac{1}{2}\frac{r^2}{a^2}\right)e^{-r/a}$$

$$= \frac{a}{\sqrt{48}}\int_0^\infty \left(\alpha^4 - \frac{1}{2}\alpha^5\right)e^{-\alpha}\, d\alpha \qquad \left(\alpha = \frac{r}{a}\right)$$

$$= -3\sqrt{3}\, a$$

$$\langle \psi_{200} | \Delta V | \psi_{210}\rangle = (-e\mathcal{E}_0)(-3\sqrt{3}\, a)\frac{1}{\sqrt{3}}$$

$$\boxed{|\Delta E| = 3ae\mathcal{E}_0}$$ (see diagram next page)

(b)

$$\begin{pmatrix} 0 & |\Delta E| \\ |\Delta E| & 0 \end{pmatrix}\begin{pmatrix} \cos\eta_\pm \\ \sin\eta_\pm \end{pmatrix} = \pm|\Delta E|\begin{pmatrix} \cos\eta_\pm \\ \sin\eta_\pm \end{pmatrix}$$

$$\cos\eta_+ = \sin\eta_+ = 1/\sqrt{2}$$
$$-\sin\eta_- = \cos\eta_- = 1/\sqrt{2}$$
mixing

$n=2$

$$\psi_+ = \frac{1}{\sqrt{2}}(\psi_{200} + \psi_{210})$$

$n=2,\ l=1,\ m=\pm 1$ unaffected $\quad |\Delta E| = 3aeE_0$

$$\psi_- = \frac{1}{\sqrt{2}}(\psi_{200} - \psi_{210})$$

(c) $$\psi(0) = \frac{1}{\sqrt{2}}(\psi_+ + \psi_-)$$

$$\psi(t) = \frac{1}{\sqrt{2}}\left(e^{-i|\Delta E|t/\hbar}\psi_+ + e^{i|\Delta E|t/\hbar}\psi_-\right)e^{-iE_2t/\hbar}$$

$$= e^{-iE_2t/\hbar}\left(\psi_{200}\cos\left(\frac{|\Delta E|t}{\hbar}\right) - i\psi_{210}\sin\left(\frac{|\Delta E|t}{\hbar}\right)\right)$$

$$\boxed{\therefore \text{Prob.}(t) = |\langle\psi_{200}|\psi(t)\rangle|^2 = \cos^2\left(\frac{|\Delta E|t}{\hbar}\right)}$$

Of course this is for $t \geq 0$. For $t \leq 0$ Prob. $= 1$.

Jan 2008.

[II-3] Solutions

(a) We assume the magnetic moment is a vector parallel to the spin with a moment $\vec{\mu}_j = \mu_0 \vec{S}_j$ where μ_0 is a constant. Then we write the Hamiltonian

$$H = \frac{\mu_0^2}{a^3}\left[\vec{S}_1 \cdot \vec{S}_2 - 3 S_{1z} S_{2z}\right],$$ with the second term containing only S_{jz} component since the vector $\vec{a}$ is along the z-direction

(b) We write

$$\vec{S} = \vec{s}_1 + \vec{s}_2$$

$$\vec{S} \cdot \vec{S} = S(S+1) = (s_1 + s_2)\cdot(s_1 + s_2)$$

$$S(S+1) = 2s(s+1) + 2\,\vec{s}_1 \cdot \vec{s}_2\,, \quad S_z = s_{1z} + s_{2z}$$

$$S_z^2 = s_{1z}^2 + s_{2z}^2 + 2 s_{1z} s_{2z}\,,$$ now for $s = \frac{1}{2}$, we have $s(s+1) = 3/4$, $S_{jz}^2 = 1/4$, so we may write

$\frac{l.c.}{S}$ $\rightarrow$ $$\vec{S}_1 \cdot \vec{S}_2 = \frac{1}{2}\, S(S+1) - 3/4\,, \quad S_{1z} S_{2z} = \frac{1}{2} S_z^2 - 1/4$$

$$H = E_0\left[S(S+1) - 3S_z^2\right] \text{ and } E_0 = \mu_0^2/2a^3.$$

(c) The addition of two angular momenta with $s = 1/2$ gives values of S which are 0 or 1: For $S = 1$ there are three possible eigenvalues of $S_z = (-1, 0, 1)$, which give an energy of $E_0(-1, 2, -1)$.

[II-3] Solution-continued.

For $S' = 0$ there is one eigenvalue of $S_z = 0$, and this state has zero energy

(II-4) Soln.

(a) $$f(\vec{k}',\vec{k}) = -\frac{m}{2\pi\hbar^2}\int d^3x\, e^{i\vec{q}\cdot\vec{x}}\, V(\vec{x}) \quad , \quad \vec{q} = \vec{k} - \vec{k}'$$

$$\int d^3x\, e^{i\vec{q}\cdot\vec{x}}\, V(\vec{x}) = \sum_{\vec{n}} e^{i\vec{q}\cdot\vec{n}a}$$

$$\boxed{\therefore f(\vec{k}',\vec{k}) = \sum_{n_1=-M}^{M}\sum_{n_2=-M}^{M}\sum_{n_3=-N}^{N}\left(\frac{-m}{2\pi\hbar^2}\right) e^{ia\vec{n}\cdot(\vec{k}-\vec{k}')}}$$

(b) $$\sum_{j=1}^{n} z^j = \frac{z(1-z^n)}{1-z} \quad \Longrightarrow$$

$$\sum_{j=-n}^{n} z^j = z^0 + \sum_{j=1}^{n} z^j + \sum_{j=1}^{n} (z^{-1})^j$$

$$= \frac{z^{-n}(1-z^{2n+1})}{1-z}$$

$$\boxed{\therefore f(\vec{k}',\vec{k}) \equiv f(\vec{q}) = \prod_{i=1,2}\left(\frac{z_i^{-M}(1-z_i^{2M+1})}{1-z_i}\right) \times \left(\frac{z_3^{-N}(1-z_3^{2N+1})}{1-z_3}\right)\left(\frac{-m}{2\pi\hbar^2}\right) \qquad z_i = e^{iaq_i}.}$$

(c) zero scattering amplitude if $1 - z_3^{2N+1} = 0$

$$\Longrightarrow \boxed{q_3 = \frac{2\pi}{a}\frac{n}{2N+1} \quad , \quad n = 1, 2, 3, \ldots}$$

$$2ka\sin^2\frac{\theta}{2} = ka(1-\cos\theta) = \frac{2\pi n}{2N+1}$$

$$1-\cos\theta = \frac{1}{ka}\frac{2\pi n}{2N+1}$$

$$\cos\theta = 1 - \frac{1}{ka}\frac{2\pi n}{2N+1}$$

$$\boxed{\theta = \cos^{-1}\left(1 - \frac{1}{ka}\frac{2\pi n}{2N+1}\right) \quad (n = 1, 2, \ldots)}$$

restriction: $\dfrac{2\pi n}{ka(2N+1)} \leq 2$

$$\Rightarrow \boxed{n \leq \frac{ka(2N+1)}{\pi}}$$

Typically $ka \gg 1$, $N \gg 1$ & many diffraction minima occur.

[II-5] <u>Solutions:</u>

Let the region of localization of the first and second electron be of the dimensions r_1, and r_2. Using the Heisenberg relations we have then for the momenta of the electrons

$$p_1 \sim \hbar/r_1 \quad , \quad p_2 \sim \frac{\hbar}{r_2} \quad ,$$ so that the

kinetic energy is of the order of magnitude

$$\frac{\hbar^2}{2\mu}\left(\frac{1}{r_1^2} + \frac{1}{r_2^2}\right).$$

The potential energy of the interaction of the electron with a nucleus of charge Z is given by

$$-Ze^2\left(\frac{1}{r_1} + \frac{1}{r_2}\right)$$ and the

mutual interaction energy of the electron is equal to $e^2/(r_1+r_2)$. To find the energy of the ground state, we look for the minimum of the total energy,

$$E(r_1, r_2) = \frac{\hbar^2}{2\mu}\left(\frac{1}{r_1^2} + \frac{1}{r_2^2}\right) - Ze^2\left(\frac{1}{r_1} + \frac{1}{r_2}\right) + \frac{e^2}{(r_1+r_2)}$$

The minimum is realized for the values

Jan 2008.

[II-5] Solution - continued.

$$r_1 = r_2 = \frac{\hbar^2}{\mu e^2} \frac{1}{(Z-\frac{1}{4})}$$, therefore

the ground state energy of an ion with two electrons and nuclear charge Z is equal to

$$E \sim -(Z-\tfrac{1}{4})^2 \frac{\mu e^4}{\hbar^2} = -2(Z-\tfrac{1}{2})^2 Ry$$

$$Ry = \frac{1}{2} \frac{\mu e^4}{\hbar^2} = 13.5 \text{ eV}$$

Now calculating for Li^+ gives

$$E_{Calc} = -15.12 \text{ eV} \leftarrow \text{from above}$$

$$E_{exp} = -14.56 \text{ eV} \leftarrow \text{from measurement}$$

II-6

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\psi + V(\vec{x})\psi \Rightarrow \frac{\partial\psi}{\partial t} - \frac{i\hbar}{2m}\Delta + \frac{V(\vec{x})\psi}{i\hbar} = 0$$

$$-i\hbar\frac{\partial\psi^*}{\partial t} = -\frac{\hbar^2}{2m}\Delta\psi^* + V(\vec{x})\psi^* \Rightarrow \frac{\partial\psi^*}{\partial t} + \frac{i\hbar}{2m}\Delta\psi^* + \frac{V(\vec{x})\psi^*}{i\hbar} = 0$$

$$\rho(\vec{x},t) = |\psi(\vec{x},t)|^2 = \psi^*(\vec{x},t)\,\psi^*(\vec{x},t)$$

$$\vec{j}(\vec{x},t) = \frac{i\hbar}{2m}(\psi\vec{\nabla}\psi^* - \psi^*\vec{\nabla}\psi)$$

$$\frac{\partial\rho}{\partial t} + \vec{\nabla}\vec{j} = 0$$

$$\frac{\partial\rho}{\partial t} = \frac{\partial}{\partial t}(\psi^*\psi) = \dot{\psi}^*\psi + \psi^*\dot{\psi}$$

$$\vec{\nabla}\cdot\vec{j} = \frac{i\hbar}{2m}\vec{\nabla}(\psi\vec{\nabla}\psi^* - \psi^*\vec{\nabla}\psi) = \frac{i\hbar}{2m}(\vec{\nabla}\psi\vec{\nabla}\psi^* + \psi\Delta\psi^* - \vec{\nabla}\psi^*\vec{\nabla}\psi - \psi^*\Delta\psi)$$

$$\vec{\nabla}\cdot\vec{j} = \frac{i\hbar}{2m}(\psi\Delta\psi^* - \psi^*\Delta\psi)$$

$$\psi\frac{\partial\psi^*}{\partial t} + \psi^*\frac{\partial\psi}{\partial t} + \frac{i\hbar}{2m}(\psi\Delta\psi^* - \psi^*\Delta\psi) = 0$$

$$\cancel{\psi\left(\frac{\partial\psi^*}{\partial t} + \Delta\psi^*\right) + \psi^*\left(\frac{\partial\psi}{\partial t} + \Delta\psi\right)\frac{i\hbar}{2m} = 0}$$

$$\psi\left(\frac{\partial\psi^*}{\partial t} + \frac{i\hbar}{2m}\Delta\psi^*\right) + \psi^*\left(\frac{\partial\psi}{\partial t} - \frac{i\hbar}{2m}\Delta\psi\right) = 0$$

$$\cancel{\psi\left(\frac{\partial\psi^*}{\partial t} + \frac{i\hbar}{2m}\Delta\psi^* + \frac{V(\vec{x})}{i\hbar}\psi^*\right)}$$

$$\cancel{+\psi^*\left(\frac{\partial\psi}{\partial t} - \frac{i\hbar}{2m}\Delta\psi + \frac{V(\vec{x})}{i\hbar}\psi\right) = 0}$$

$$\psi\left[-\frac{V(x)\psi^*}{i\hbar}\right] + \psi^*\left[\frac{V(x)\psi}{i\hbar}\right] = 0$$

$$\frac{1}{i\hbar}\Big[\left[-\psi V(x)\psi^*\right] + \left[\psi^* V(x)\psi\right]\Big] = 0$$

because $V(x)$ is a hermitian operator and $\psi(x,t)$ is an eigenfunct of H

$$\cancel{-\int\psi V(x)\psi^*\,dV + \int\psi^* V(x)\psi\,dV = 0}$$

and eigenvalues are real.

QE. PhD. Jan 2008. [redacted]

II-7 Solution

phase α : $\beta p = a + b\beta\mu$

phase γ : $\beta p = c + d(\beta\mu)^2$

at phase equilibrium:

$\beta p^{(\alpha)} = \beta p^{(\gamma)}$, $\beta\mu^{(\alpha)} = \beta\mu^{(\gamma)}$ and

$\beta^{(\alpha)} = \beta^{(\gamma)} = \beta$ Thus $a + b\beta\mu = c + d(\beta\mu)^2$

implying, $\beta\mu = \dfrac{b \pm \sqrt{b^2 - 4d(c-a)}}{2d}$

To obtain the density use the Gibbs-Duhem equation,

$d\mu = -s\,dT + v\,dp$ or:

$$\frac{1}{v} = \rho = \left(\frac{\partial p}{\partial \mu}\right)_T = \left(\frac{\partial(\beta p)}{\partial(\beta\mu)}\right)_\beta$$

so: $\rho^{(\alpha)} = b$

$\rho^{(\gamma)} = 2d\beta\mu$ and we identify the positive root as the physical root to the quadratic equation. Hence,

$\rho^{(\gamma)} - \rho^{(\alpha)} = \sqrt{b^2 + 4d(a-c)}$ and

$$\beta p_{\text{trans.}} = a + \frac{b}{2d}\left[\sqrt{b^2 + 4d(a-c)}\right].$$

II - 8

$$F(T,V,N) = F^{ideal}(T,V,N) - \frac{2}{3} N e^2 \left(\frac{4\pi N e^2}{V k_B T}\right)^{1/2}$$

a) $$P = -\left(\frac{\partial F}{\partial V}\right)_{T,N} = \underbrace{-\left(\frac{\partial F^{ideal}}{\partial V}\right)_{T,N}}_{P^{ideal}(T,V)} - \frac{2}{3} N e^2 \left(+\frac{1}{2}\frac{1}{V}\right)\left(\frac{4\pi N e^2}{V k_B T}\right)^{1/2}$$

$$= \frac{N k_B T}{V} - \frac{1}{3}\frac{N e^2}{V}\left(\frac{4\pi N e^2}{V k_B T}\right)^{1/2}$$

b) $F = E - TS$

$E = F + TS$ and $S = -\left(\frac{\partial F}{\partial T}\right)_{V,N}$

$$S = -\left(\frac{\partial F}{\partial T}\right)_{V,N} = -\left(\frac{\partial F^{ideal}}{\partial T}\right)_{V,N} - \frac{2}{3} N e^2 \left(+\frac{1}{2}\frac{1}{T}\right)\left(\frac{4\pi N e^2}{V k_B T}\right)^{1/2}$$

$$= S^{ideal} - \frac{1}{3}\frac{N e^2}{T}\left(\frac{4\pi N e^2}{V k_B T}\right)^{1/2}$$

$$E = F + TS = F^{ideal} - \frac{2}{3} N e^2 \left(\frac{4\pi N e^2}{V k_B T}\right)^{1/2} + T S^{ideal} - \frac{1}{3} N e^2 \left(\frac{4\pi N e^2}{V k_B T}\right)^{1/2}$$

$$= \underbrace{F^{ideal} + T S^{ideal}}_{E^{ideal}} - N e^2 \left(\frac{4\pi N e^2}{V k_B T}\right)^{1/2} =$$

$$= E^{ideal} - N e^2 \left(\frac{4\pi N e^2}{V k_B T}\right)^{1/2} = \frac{3}{2} N k_B T - N e^2 \left(\frac{4\pi N e^2}{V k_B T}\right)^{1/2}$$

c) $$C_V = \left(\frac{\partial E}{\partial T}\right)_{V,N} = \frac{3}{2} N k_B + \frac{N e^2}{2T}\left(\frac{4\pi N e^2}{V k_B T}\right)^{1/2}$$

other possible way to obtain C_V:

$$C_V = \left(\frac{\delta Q}{\delta T}\right)_{V,N} = T\left(\frac{\partial S}{\partial T}\right)_{V,N} = -T\left(\frac{\partial^2 F}{\partial T^2}\right)_{V,N} =$$

$$= -T\left\{\left(\frac{\partial^2 F^{ideal}}{\partial T^2}\right)_{V,N} - \frac{2}{3}\cdot\frac{1}{2}\cdot\frac{3}{2}\,\frac{N e^2}{T^2}\left(\frac{4\pi N e^2}{V k_B T}\right)^{1/2}\right\} =$$

$$= C_V^{ideal} + \frac{N e^2}{2T}\left(\frac{4\pi N e^2}{V k_B T}\right)^{1/2} = \frac{3}{2} N k_B + \frac{N e^2}{2T}\left(\frac{4\pi N e^2}{V k_B T}\right)^{1/2}$$

(II-9) Soln.

(a) $$Z = \frac{1}{h^3}\int d^3p\, d^3x \;\; e^{-\beta E(\vec{p},\vec{x})}$$

$$E(\vec{p},\vec{x}) = \frac{\vec{p}^{\,2}}{2m} + mgz$$

Z_1 = 1-particle partition fcn.

$$= \frac{L^2}{h^3}\left(\int_0^L dz\, e^{-\beta mgz}\right)\left(\int dp\, e^{-\beta p^2/2m}\right)^3$$

$$\boxed{\begin{aligned} Z_1 &= \frac{L^2}{h^3}\cdot(2\pi mkT)^{3/2}\,\frac{kT}{mg}\left(1-e^{-mgL/kT}\right) \\ Z_N &= Z_1^N \end{aligned}}$$

(b) $L = V^{1/3}$, $\quad F = -kT\ln Z_N = -NkT\ln Z_1$

$$P = -\left(\frac{\partial F}{\partial V}\right)_T = NkT\,\frac{\partial Z_1/\partial V}{Z_1}$$

$$\boxed{P = NkT\left(\frac{2}{3V} + \frac{mg}{3kT\,V^{2/3}}\;\frac{e^{-mgV^{1/3}/kT}}{1-e^{-mgV^{1/3}/kT}}\right)}$$

(c) $$1 - e^{-mgV^{1/3}/kT} \simeq \frac{mgV^{1/3}}{kT}$$

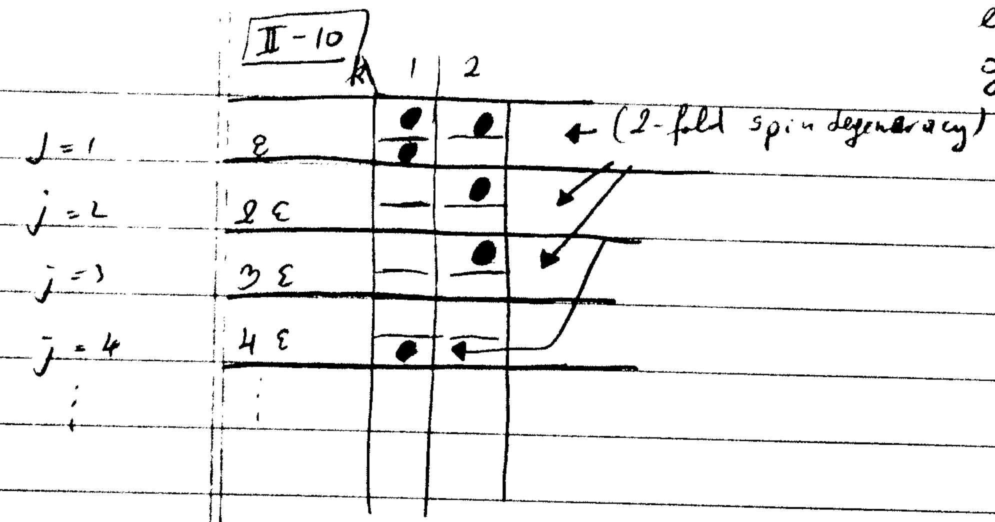
$$P \simeq NkT\left(\frac{2}{3V} + \frac{mg}{3kTV^{2/3}}\cdot\frac{kT}{mgV^{1/3}}\right)$$

$$= \frac{NkT}{V} \qquad //$$

$\boxed{\text{II}-10}$

electrons: $s = 1/2$ (Fermions,

$g_j = 2$ for each j



$$\boxed{\begin{aligned} E &= \sum_j n_j \varepsilon_j = 6 \\ N &= \sum_j n_j = 3 \end{aligned}}$$

$\varepsilon_j = j\,\varepsilon$ energy

Only 2 macrostates satisfy the constraint.

$\underline{k=1)}$: $n_1 = 2,\ n_4 = 1;\ n_j = 0$ for all other j

$\underline{k=2)}$: $n_1 = 1,\ n_2 = 1,\ n_3 = 1,\ n_j = 0$ for all other j

<u>indistinguishable particles – fermions</u>

number of microstates associated with macrostate k:

a)

$w_1 = 1 \cdot 2 = 2$

$w_2 = 2 \cdot 2 \cdot 2 = 2^3 = 8$

$\Omega = \sum_k w_k = 10 \implies \boxed{S(N=3, E=6\varepsilon) = k_B \ln(10)}$

b) $\overline{n}_1 = 2 \cdot \frac{w_1}{\Omega} + 1 \cdot \frac{w_2}{\Omega} = 2 \cdot \frac{2}{10} + 1 \cdot \frac{8}{10} = \frac{12}{10} = \frac{6}{5}$

$\overline{n}_2 = 0 \cdot \frac{w_1}{\Omega} + 1 \cdot \frac{w_2}{\Omega} = \frac{8}{10} = \frac{4}{5}$

$\overline{n}_3 = 0 \cdot \frac{w_1}{\Omega} + 1 \cdot \frac{w_2}{\Omega} = \frac{8}{10} = \frac{4}{5}$

$\overline{n}_4 = 1 \cdot \frac{w_1}{\Omega} + 0 \cdot \frac{w_2}{\Omega} = \frac{2}{10} = \frac{1}{5}$

$\overline{n}_j = 0$ for $j > 4$

(Check: $\overline{n}_1 + \overline{n}_2 + \overline{n}_3 + \overline{n}_4 + \ldots$

$= n_1 + n_2 + n_3 + n_4 + \ldots$