Solutions

Physics PhD Qualifying exam
Part I January 2012

[I-1] **[10]**

Imagine a narrow shaft connecting two diagonally opposite points on the surface of the Earth (going through the center of the Earth). A small object is dropped from rest into the shaft. Ignore air resistance, melting temperatures, etc. The total mass of the Earth is M, the radius of the Earth is R, and the gravitational constant is G. Assume that the Earth's mass is homogeneously distributed in its volume.

By obtaining the equation of motion of the object (along the shaft), show that the resulting motion is oscillatory and obtain the period of the oscillations. You must express your answer in terms of M, R, and G.

[I-2] **[3,4,3]**

Let the Lagrangian of two interacting point particles be

$$L = \frac{1}{2}m_1\dot{\vec{r}}_1^{\,2} + \frac{1}{2}m_2\dot{\vec{r}}_2^{\,2} - \frac{\alpha}{|\vec{r}_1 - \vec{r}_2|} \quad .$$

(a) Using the relative position vector $\vec{r} = \vec{r}_1 - \vec{r}_2$, rewrite the Lagrangian in spherical coordinates in the *center of mass* reference frame.

(b) Write down the Euler-Lagrange equations for the two angles and identify from these equations a conserved quantity.

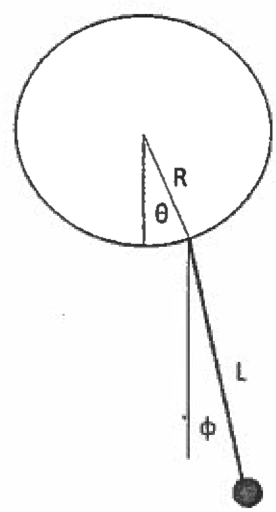
(c) Prove that the motion is confined to a plane.

[I-3] [3,7]

A pendulum bob with mass m hangs from a disk free to rotate about its center with moment of inertia I. The radius of the disk is R and the length of the string holding the bob is L, as shown in the figure below.

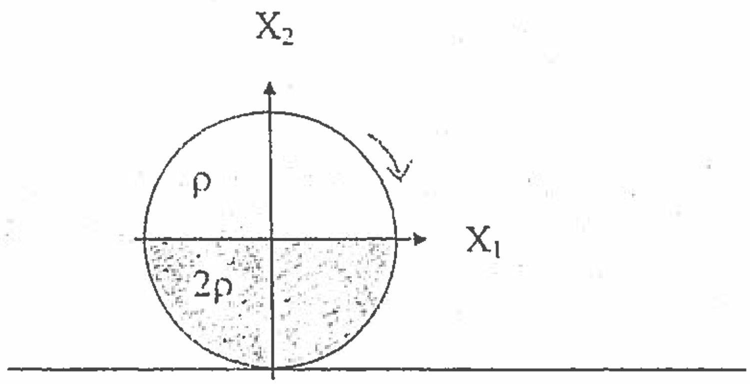
(a) Using the coordinates θ and φ as shown, construct the Lagrangian and then reduce it to a quadratic form for small angles (i.e., the equations of motion would be linear). Hints: The Cartesian coordinates of the bob are $x = R\sin\theta + L\sin\phi$ and $y = -R\cos\theta - L\cos\phi$. The Lagrangian to start from is $L = \frac{1}{2}I\dot{\theta}^2 + \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{y}^2 - mgy$.

(b) Calculate the normal mode frequencies.



[I-4] [10]

Consider a thin disk composed of two homogeneous halves connected along the diameter of the disk. One half of the disk has density ρ and the other has density 2ρ. The total mass of the disk is M and its radius is R. Find the expression for the Lagrangian in terms of the relevant angular rotation variable ϑ (measured from the vertical direction) when the *disk rolls without slipping* along a horizontal surface. The rotation takes place in the plane of the disk.



[I-5] [10]

In reference frame S, a train is moving to the $+x$ direction with velocity $\frac{2}{3}c$, where c is the speed of light. Also as observed in reference frame S, a car is moving to the $-x$ direction with velocity $-\frac{1}{3}c$.

What is the velocity of the car as observed by a passenger sitting on the train?
(You must use special relativity.)

[I-6] [10]

An infinitely long solid cylinder, centered along the z-axis, has a cross sectional radius a and a uniform volume charge density ρ_{ch}. Calculate the scalar potential ϕ everywhere (inside and outside the cylinder) with the convention that the scalar potential is zero at the center of the cylinder.

[I-7] [10]

Consider the following electromagnetic field:

$$\vec{E}(\vec{x},t)=A\cos\left(\vec{k}\bullet\vec{x}-\omega t\right)\hat{x}+B\sin\left(\vec{k}\bullet\vec{x}-\omega t\right)\hat{y}$$
$$\vec{B}(\vec{x},t)=C\sin\left(\vec{k}\bullet\vec{x}-\omega t\right)\hat{x}+D\cos\left(\vec{k}\bullet\vec{x}-\omega t\right)\hat{y}$$

with $A, B, C, D =$ constants, unit vectors $\hat{x}$, $\hat{y}$, $\hat{z}$, $\vec{k}=k\,\hat{z}$, and $\vec{x}=x_1\,\hat{x}+x_2\,\hat{y}+x_3\,\hat{z}$.

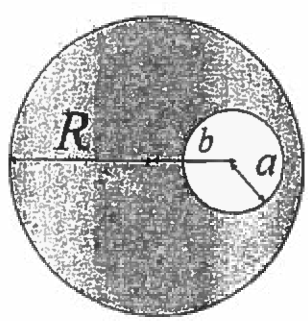
Constants A and B are given.

(a) Write down Maxwell's equations in vacuum.

(b) Determine C and D such that $\vec{E}(\vec{x},t)$ and $\vec{B}(\vec{x},t)$ fulfill Maxwell's equation in vacuum.

[I-8] [10]

A long straight conductor with a circular cross section of radius R carries a current I. Inside the conductor is a cylindrical hole of radius a whose axis is parallel to the axis of the conductor but offset a distance b from the center of the conductor. The current of the conductor is uniformly distributed across the cross section of the conductor and is directed parallel to the cylinder axis. Find the magnetic field everywhere **outside** the conductor.



[I-9] [4,4,2]

(a) The vector potential due to a harmonic electric dipole $\boldsymbol{p} = p_0\hat{\mathbf{z}}e^{-i\omega t}$ is given by:

$$\boldsymbol{A} = -\frac{i\mu_0\omega p_0\hat{\mathbf{z}}}{4\pi r}e^{i(kr-\omega t)}$$

Show that the electric field and magnetic induction in the radiation zone $kr \gg 1$ are given by:

$$\boldsymbol{E} = -\frac{p_0 k^2}{4\pi\epsilon_0 r}\sin\theta\, e^{i(kr-\omega t)}\hat{\boldsymbol{\theta}}$$

$$\boldsymbol{B} = -\frac{p_0\mu_0 k\omega}{4\pi r}\sin\theta\, e^{i(kr-\omega t)}\hat{\boldsymbol{\varphi}}$$

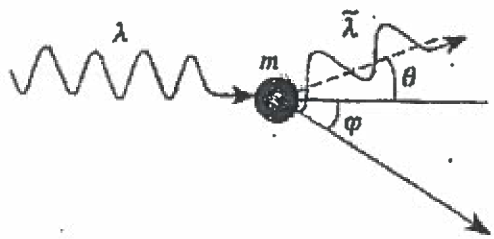
Hint: Determine $\boldsymbol{B}$ from $\boldsymbol{A}$. Then use one of Maxwell's equations and the fact that $\boldsymbol{E} \sim e^{-i\omega t}$ to determine $\boldsymbol{E}$ from $\boldsymbol{B}$. You will also need the following formula:

$$\nabla\times\boldsymbol{A} = \frac{\hat{r}}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta\, A_\varphi) - \frac{\partial A_\theta}{\partial\varphi}\right] + \frac{\hat{\theta}}{r}\left[\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\varphi} - \frac{\partial}{\partial r}(rA_\varphi)\right] + \frac{\hat{\varphi}}{r}\left[\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial\theta}\right]$$

(b) Compute the time averaged differential power $dP/d\Omega$ radiated into an element of solid angle, as a function of θ.

(c) If the frequency is doubled, by what factor is the power radiated increased?

[I-10] [10]
In the Compton effect, a $\gamma - ray$ photon of wavelength λ strikes a free but initially stationary, electron of mass m. The photon is scattered at an angle θ, and its scattered wavelength is $\tilde{\lambda}$. The electron recoils at an angle φ (see figure below).



(a) Write the relativistic equations for momentum and energy conservation.
(b) Find an expression for the change $\tilde{\lambda} - \lambda$ in the photon wavelength for the special case $\theta = \pi/2$.

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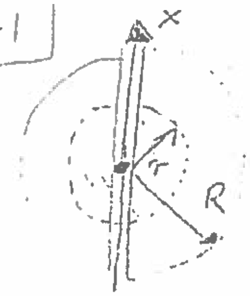


QE, January 2012
Part I – Solutions

Using Shell Theorem:

$$F(r) = -G\frac{M_{encl}(r)\,m}{r^2} \qquad M_{encl}(r) = M\frac{r^3}{R^3}$$

$$F(r) = -G\frac{M\frac{r^3}{R^3}\,m}{r^2} = -G\frac{Mm}{R^3}\cdot r$$

Placing the axis of motion (e.g., x) along the shaft, we find the following equation of motion:

$$m\ddot{x} = F(x) = -G\frac{Mm}{R^3}x \qquad \text{(force always points toward origin (center of Earth))}$$

$$\cancel{m}\ddot{x} = -G\frac{M\cancel{m}}{R^3}x$$

$$\boxed{\ddot{x} = -\frac{GM}{R^3}x} \leftarrow \text{equation of motion for simple harmonic oscillator with frequency}$$

$$\omega^2 = \frac{GM}{R^3}$$

$$T = \frac{2\pi}{\omega} = \boxed{2\pi\sqrt{\frac{R^3}{GM}}}$$

Solution $\boxed{\text{I.2}}$

a) $\vec{R}_{cm} = \dfrac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{m_1 + m_2}$

$R_{cm} = 0$ in cm reference.

$$\vec{r} = r_1 - r_2 + \left(\frac{m_1 + m_2}{m_2}\right) R$$

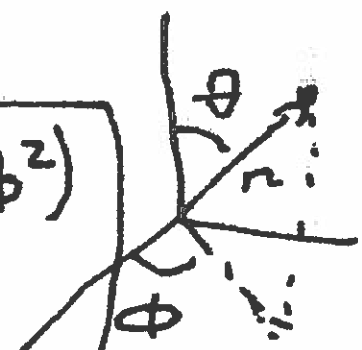
$$= \frac{m_1 + m_2}{m_2} \vec{r}_1 - \frac{m_1 + m_2}{m_1} \vec{r}_2$$

In the cm frame

$$r_1 = \frac{m_2}{m_1 + m_2} \vec{r} \qquad r_2 = \frac{m_1}{m_1 + m_2} \vec{r}$$

$$\boxed{L = \mu \dot{\vec{r}}^2 - \frac{\alpha}{|r|}}$$ with $\mu = \dfrac{m_1 m_2}{m_1 + m_2}$

In spherical coordinates



$$\boxed{L = \frac{1}{2}\mu \dot{r}^2 + \frac{1}{2}\mu r^2 (\dot{\theta}^2 + \sin^2\theta \, \dot{\phi}^2) - \frac{\alpha}{r}}$$

(I-2, cont)

Euler-Lagrange

i) $$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta}\right) - \frac{\partial L}{\partial \theta} = 0$$

$$\to 0 = 2\mu r \dot r \dot\theta + \mu r^2 \left(\ddot\theta - \sin\theta\cos\theta\,\dot\phi^2\right)$$

ii) $$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\phi}\right) - \underbrace{\cancel{\frac{\partial L}{\partial \phi}}}_{0} = 0$$

$$\frac{d}{dt}\Big(\underbrace{\mu r^2 \sin^2\theta\,\dot\phi}_{\text{conserved quantity}}\Big) = 0$$

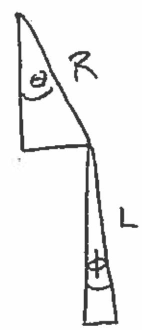
B) Choose coordinates such that $\dot\phi(0)$ is orthogonal to $\underbrace{\vec r(0) \times \dot{\vec r}(0)}_{\vec J(0) \equiv \text{angular momentum}}$, $\theta \neq 0, \pi$.

then $\sin^2(\theta(t))\,\dot\phi(t) = 0 \to \dot\phi = 0$

ϕ = constant $\to$ plane.

$\boxed{I-3}$

Problem I-3



$$x = R\sin\theta + L\sin\phi$$

$$y = -R\cos\theta - L\cos\phi$$

$$\dot{x} = R\cos\theta\,\dot\theta + L\cos\phi\,\dot\phi$$

$$\approx R\dot\theta + L\dot\phi$$

$$\dot{y} = R\sin\theta\,\dot\theta + L\sin\phi\,\dot\phi$$

$$\approx 0$$

since $\mathcal{O}(\theta\dot\theta)$, $\mathcal{O}(\phi\dot\phi)$.

$$V = mgy$$

$$\approx +\tfrac{1}{2}mg\left(R\theta^2 + L\phi^2\right) - (R+L)mg$$

$$T = \tfrac{1}{2}I\dot\theta^2 + \tfrac{1}{2}m\dot{x}^2 + \tfrac{1}{2}m\dot{y}^2$$

$$\approx \tfrac{1}{2}I\dot\theta^2 + \tfrac{1}{2}m\left(R\dot\theta + L\dot\phi\right)^2$$

$$L \approx \tfrac{1}{2}I\dot\theta^2 + \tfrac{1}{2}m\left(R\dot\theta + L\dot\phi\right)^2$$

$$-\tfrac{1}{2}mg\left(R\theta^2 + L\phi^2\right) + (R+L)mg$$

$$\frac{\partial L}{\partial\theta} = -mgR\theta \qquad \frac{\partial L}{\partial\phi} = -mgL\phi$$

$$\frac{\partial L}{\partial\dot\theta} = I\dot\theta + mR\left(R\dot\theta + L\dot\phi\right)$$

(I-3, cont)

$$\frac{\partial L}{\partial \dot{\phi}} = mL(R\dot{\theta} + L\dot{\phi})$$

EOM:

$$I\ddot{\theta} + mR^2\ddot{\theta} + mRL\ddot{\phi} + mgR\theta = 0$$

$$mLR\ddot{\theta} + mL^2\ddot{\phi} + mgL\phi = 0$$

$$(\theta, \phi) = (c_1, c_2)e^{i\omega t}$$

$$(\ddot{\theta}, \ddot{\phi}) = -\omega^2 (c_1, c_2) e^{i\omega t}$$

$$(I + mR^2)(-\omega^2)c_1 - \omega^2 mRL c_2 + mgRc_1 = 0$$

$$(mLR)(-\omega^2)c_1 - \omega^2 mL^2 c_2 + mgLc_2 = 0$$

$$\begin{vmatrix} -\omega^2(I+mR^2) + mgR & -\omega^2 mRL \\ -\omega^2 mLR & -\omega^2 mL^2 + mgL \end{vmatrix} = 0$$

$$\omega^4 mL^2(I+mR^2) - \omega^2(mL^2 mgR + (I+mR^2)mgL)$$
$$+ m^2g^2RL - \omega^4 m^2R^2L^2 = 0$$

(I-3, cont.)

$$\omega^4 mL^2 I - \omega^2 (m^2 L^2 R g + I m g L + m^2 R^2 L g)$$
$$+ m^2 g^2 R L = 0$$

$$\omega_{\pm}^2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$b = -(m^2 g L^2 R + m^2 g R^2 L + m g I L)$$

$$a = mL^2 I$$

$$c = m^2 g^2 R L$$

Solution I-4

The center of mass of the disk is $(0, \bar{x}_2)$, where

$$\bar{x}_2 = \frac{\rho}{M}\left[2\int_{\substack{\text{lower}\\\text{semicircle}}} x_2\, dx_1\, dx_2 + \int_{\substack{\text{upper}\\\text{semicircle}}} x_2\, dx_1\, dx_2 \right]$$

$$= \frac{\rho}{M}\left[\int_0^R \int_0^\pi (r\sin\theta)\cdot r\,dr\,d\theta + 2\int_0^R \int_\pi^{2\pi} (r\sin\theta)\cdot r\,dr\,d\theta \right]$$

$$= -\frac{2}{3}\frac{\rho R^3}{M} \qquad (1)$$

Now, the mass of the disk is

$$M = \rho\cdot\frac{1}{2}\pi R^2 + 2\rho\cdot\frac{1}{2}\pi R^2$$

$$= \frac{3}{2}\rho\pi R^2 \qquad (2)$$

so that

$$\bar{x}_2 = -\frac{4}{9\pi}R \qquad (3)$$

The direct calculation of the rotational inertia with respect to an axis through the center of mass is tedious, so we first compute I with respect to the x_3-axis and then use Steiner's theorem.

$$I_3 = \rho\left[\int_0^R \int_0^\pi r^2\cdot r\,dr\,d\theta + 2\int_0^R \int_\pi^{2\pi} r^2\cdot r\,dr\,d\theta \right]$$

$$= \frac{3}{4}\pi\rho R^4 = \frac{1}{2}M R^2 \qquad (4)$$

Then,

$$I_0 = I_3 - M\bar{x}_2^2$$

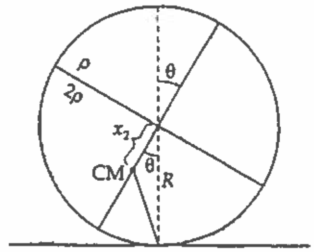
$$= \frac{1}{2}M R^2 - M\cdot\frac{16}{81\pi^2}R^2$$

$$= \frac{1}{2}M R^2\left[1 - \frac{32}{81\pi^2}\right] \qquad (5)$$

When the disk rolls without slipping, the velocity of the center of mass can be obtained as follows:

Thus

(I-4, cont.)



$$x_{CM} = R\theta - |\overline{x}_2| \sin\theta$$

$$y_{CM} = R - |\overline{x}_2| \cos\theta$$

$$\dot{x}_{CM} = R\dot{\theta} - |\overline{x}_2|\, \dot{\theta}\cos\theta$$

$$\dot{y}_{CM} = |\overline{x}_2|\, \dot{\theta}\sin\theta$$

$$\left(\dot{x}_{CM}^2 + \dot{y}_{CM}^2\right) = V^2 = R^2\dot{\theta}^2 + \overline{x}_2^2\,\dot{\theta}^2 - 2\dot{\theta}^2 R|\overline{x}_2| \cos\theta$$

$$V^2 = a^2\dot{\theta}^2 \tag{6}$$

where

$$a = \sqrt{R^2 + \overline{x}_2^2 - 2R\,|\overline{x}_2| \cos\theta} \tag{7}$$

Using (3), a can be written as

$$a = R\sqrt{1 + \frac{16}{81\pi^2} - \frac{8}{9\pi}\cos\theta} \tag{8}$$

The kinetic energy is

$$T = T_{trans} + T_{rot}$$

$$= \frac{1}{2} M\, v^2 + \frac{1}{2} I_0\dot{\theta}^2 \tag{9}$$

Substituting and simplifying yields

$$T = \frac{1}{2} M\, R^2\dot{\theta}^2 \left[\frac{3}{2} - \frac{8}{9\pi}\cos\theta\right] \tag{10}$$

The potential energy is

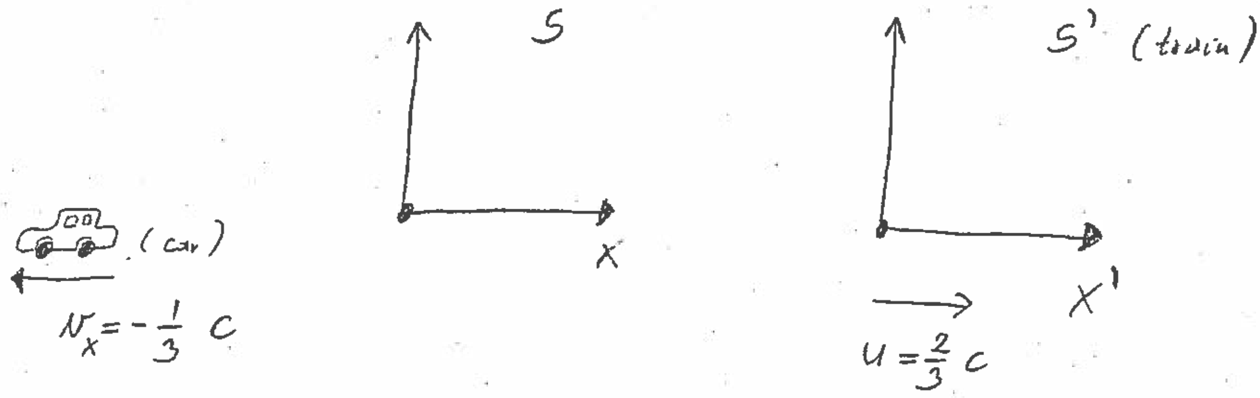
$$U = M\, g\left[\frac{1}{2} R + \overline{x}_2 \cos\theta\right]$$

$$= \frac{1}{2} M\, gR\left[1 - \frac{8}{9\pi}\cos\theta\right] \tag{11}$$

Thus the Lagrangian is

$$\boxed{L = \frac{1}{2} M\, R\left[R\dot{\theta}^2\left[\frac{3}{2} - \frac{8}{9\pi}\cos\theta\right] - g\left[1 - \frac{8}{9\pi}\cos\theta\right]\right]} \tag{12}$$

I-5



Relativistic velocity transformation:

$$\boxed{v'_x = \frac{v_x - u}{1 - \frac{u v_x}{c^2}}} = \frac{-\frac{1}{3}c - \frac{2}{3}c}{1 + \frac{\frac{1}{3}c\,\frac{2}{3}c}{c^2}} =$$

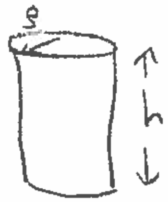
$$= \frac{-c}{1 + \frac{2}{9}} = -\frac{c}{11/9} = \boxed{-\frac{9}{11}c}$$

Problem I-6

Use cylindrical coordinates ρ, θ, z.

Then $\vec{E} = E(\rho)\hat{\rho}$ by symmetry.

By Gauss' Law,



$$2\pi\rho h E(\rho) = \frac{1}{\epsilon_0}\pi\rho^2 h \rho_{ch}$$

$$E(\rho) = \frac{\rho \rho_{ch}}{2\epsilon_0} \quad \text{inside}$$

$$2\pi\rho h E(\rho) = \frac{1}{\epsilon_0}\pi a^2 h \rho_{ch}$$

$$E(\rho) = \frac{a^2 \rho_{ch}}{2\rho\epsilon_0} \quad \text{outside}$$

$$\phi = -\int_0^\rho d\rho' E(\rho') = -\frac{1}{4\epsilon_0}\rho^2 \rho_{ch} \quad \text{inside}$$

$$\phi = -\frac{1}{4\epsilon_0}a^2\rho_{ch} - \int_a^\rho d\rho' E(\rho')$$

$$= -\frac{a^2\rho_{ch}}{4\epsilon_0} - \frac{a^2\rho_{ch}}{2\epsilon_0}\ln\left(\frac{\rho}{a}\right) \quad \text{outside.}$$

I-7 [redacted] I-7

Maxwell's equations problem PROBLEM # 7 Part I

$$\vec{E}(\vec{x},t) = \alpha\,\vec{e}_1 \cos(\vec{k}\cdot\vec{x}-\omega t) + \beta\,\vec{e}_2 \sin(\vec{k}\cdot\vec{x}-\omega t)$$

$$\vec{B}(\vec{x},t) = \frac{1}{c}\left\{ \gamma\,\vec{e}_1 \sin(\vec{k}\cdot\vec{x}-\omega t) + \delta\,\vec{e}_2 \cos(\vec{k}\cdot\vec{x}-\omega t)\right\}$$

$$\vec{k}\cdot\vec{x} = k x_3$$

a) Maxwellgleichungen, mit $\varrho = 0$; $\vec{j} = 0$

$$\vec{\nabla}\cdot\vec{E} = 0 \qquad \vec{\nabla}\times\vec{E} = -\dot{\vec{B}}$$

$$\vec{\nabla}\cdot\vec{B} = 0 \qquad \vec{\nabla}\times\vec{B} = \mu_0\varepsilon_0\,\dot{\vec{E}}$$

$$\vec{\nabla}\times\vec{E} = \begin{pmatrix}\frac{\partial}{\partial x_1}\\ \frac{\partial}{\partial x_2}\\ \frac{\partial}{\partial x_3}\end{pmatrix} \times \begin{pmatrix}\alpha\cos(kx_3-\omega t)\\ \beta\sin(kx_3-\omega t)\\ 0\end{pmatrix}$$

$\omega = k\cdot c$

$$= -\frac{\partial}{\partial x_3}\left(\beta\sin(kx_3-\omega t)\right)\vec{e}_1 + \frac{\partial}{\partial x_3}\left(\alpha\cos(kx_3-\omega t)\right)\vec{e}_2$$

$$\vec{\nabla}\times\vec{E} = -k\beta\cos(kx_3-\omega t)\,\vec{e}_1 - k\alpha\sin(kx_3-\omega t)\,\vec{e}_2$$

$$\vec{\nabla}\times\vec{B} = \begin{pmatrix}\frac{\partial}{\partial x_1}\\ \frac{\partial}{\partial x_2}\\ \frac{\partial}{\partial x_3}\end{pmatrix} \times \begin{pmatrix}\gamma\sin(\vec{k}\cdot\vec{x}-\omega t)\\ \delta\cos(\vec{k}\cdot\vec{x}-\omega t)\\ 0\end{pmatrix}\cdot\frac{1}{c}$$

$$c\,\vec{\nabla}\times\vec{B} = -\frac{\partial}{\partial x_3}\left(\delta\cos(kx_3-\omega t)\right)\vec{e}_1 + \frac{\partial}{\partial x_3}\left(\gamma\sin(kx_3-\omega t)\right)\vec{e}_2$$

$$\vec{\nabla}\times\vec{B} = +\vec{e}_1\frac{k}{c}\delta\sin(kx_3-\omega t) + \frac{k}{c}\gamma\,\cancel{\sin}\cos(kx_3-\omega t)\vec{e}_2$$

$$\dot{\vec{B}} = \frac{1}{c}\left\{-\gamma\omega\,\vec{e}_1\cos(kx_3-\omega t) + \delta\omega\,\vec{e}_2\sin(kx_3-\omega t)\right\}$$

I-#7 cont

$$\frac{k}{c}\,\delta \sin(kx_3-\omega t) = \mu_0\varepsilon_0\,\omega\alpha \sin(kx_3-\omega t)$$

$$+\frac{k}{c}\,\gamma \cos(kx_3-\omega t) = -\mu_0\varepsilon_0\,\omega\beta \cos(kx_3-\omega t)$$

$$-k\beta \cos(kx_3-\omega t) = -\left(-\gamma\frac{\omega}{c}\cos(kx_3-\omega t)\right)$$

$$-k\alpha \sin(kx_3-\omega t) = -\left(\frac{\omega\delta}{c}\sin(kx_3-\omega t)\right)$$

$$\frac{k}{c}\,\delta = \mu_0\varepsilon_0\,\omega\alpha$$

$$\frac{k}{c}\,\delta = \frac{\omega}{c^2}\,\alpha$$

$$\frac{k}{c}\,\delta = \frac{kc}{c^2}\,\alpha \quad \Rightarrow \quad \delta = \alpha$$

$$+\frac{k}{c}\,\gamma = -\mu_0\varepsilon_0\,\omega\beta \Rightarrow \gamma = -\beta$$

$$-k\beta = \gamma\frac{\omega}{c}$$

$$\underline{\underline{-\beta = \gamma}}$$

$$k\alpha = \frac{\omega\delta}{c}$$

$$\underline{\underline{\alpha = +\delta}}$$

$$\vec{E}(\vec{x},t) = \alpha\,\vec{e}_1 \cos(kx_3-\omega t) + \beta\,\vec{e}_2 \sin(kx_3-\omega t)$$

$$c\,\vec{B}(\vec{x},t) = -\beta\,\vec{e}_1 \sin(kx_3-\omega t) + \alpha\,\vec{e}_2 \cos(kx_3-\omega t)$$

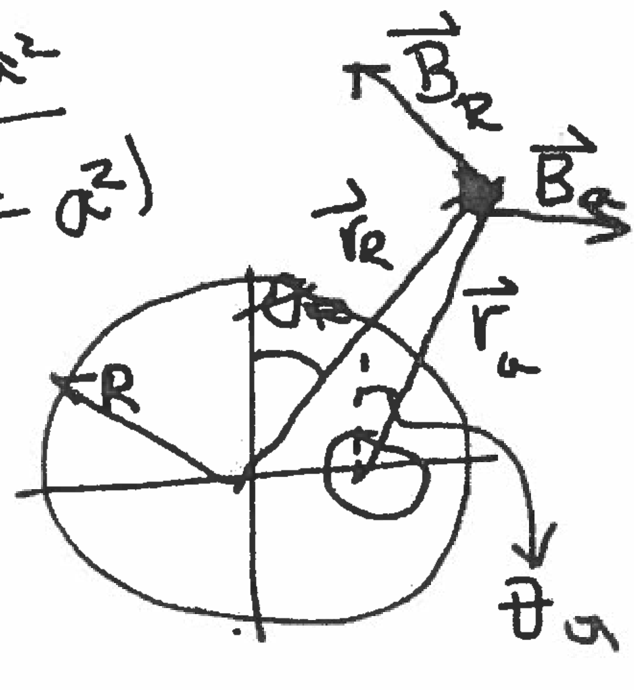
Problem $\boxed{\text{I-8}}$

Use superposition of fields produced by a uniform current along the cylinder of radius R, and the field produced by a current in the opposite direction along the cylinder of radius a.

The current density is $J = \dfrac{I}{\pi(R^2 - a^2)}$

$$I_R = J\pi R^2 = \frac{I R^2}{\pi(R^2 - a^2)}$$

$$I_a = J\pi a^2 = \frac{I a^2}{\pi(R^2 - a^2)}$$



$$B_R = \frac{\mu_0}{2\pi}\frac{I_R}{r_R}$$

$$B_a = \frac{\mu_0}{2\pi}\frac{I_a}{r_a}$$

(I-8, cont.)

$$B_{Rx} = -B_R \cos(\theta_R) = -\frac{\mu_0}{2\pi} \frac{I_R y}{x^2+y^2}$$

$$B_{Ry} = B_R \sin(\theta_R) = \frac{\mu_0}{2\pi} \frac{I_R x}{x^2+y^2}$$

$$B_{ax} = B_a \cos(\theta_a) = \frac{\mu_0}{2\pi} \frac{I_a y}{(x-b)^2+y^2}$$

$$B_{ay} = -B_a \sin(\theta_a) = \frac{\mu_0}{2\pi} \frac{I_a (x-b)}{(x-b)^2+y^2}$$

Add fields:

$$B_x = B_{Rx} + B_{ax}$$
$$= -\frac{\mu_0 I}{2\pi} \frac{1}{R^2-a^2} \left(\frac{R^2}{x^2+y^2} - \frac{a^2}{(x-b)^2+y^2} \right) y$$

$$B_y = -\frac{\mu_0 I}{2\pi} \frac{1}{R^2-a^2} \left(\frac{R^2 x}{x^2+y^2} - \frac{a^2 (x-b)}{(x-b)^2+y^2} \right)$$

$$B_z = 0$$

Problem I-9

a. In spherical coords.

$$\vec{A} = -\frac{i\mu_0 \omega p_0}{4\pi r} e^{i(kr-\omega t)}(\cos\theta\,\hat{r} - \sin\theta\,\hat{\theta})$$

Taking the curl, $\vec{B} = \vec{\nabla}\times\vec{A}$,

$$\vec{B} = -\frac{\mu_0 k^2 \omega p_0}{4\pi}\left[\frac{1}{kr} + \frac{i}{(kr)^2}\right]\sin\theta\, e^{i(kr-\omega t)}\,\hat{\varphi}$$

Then from maxwell's eq.

$$\vec{\nabla}\times\vec{B} = \mu_0\epsilon_0 \frac{\partial \vec{E}}{\partial t} = -i\frac{\omega}{c^2}\vec{E}$$

we obtain

$$\vec{E} = i\frac{c^2}{\omega}\vec{\nabla}\times\vec{B}$$

$$\vec{E} = -\frac{k^3 p_0}{4\pi\epsilon_0}\left\{\left[\frac{2i}{(kr)^2} - \frac{2}{(kr)^3}\right]\cos\theta\,\hat{r} + \left[\frac{1}{kr} + \frac{i}{(kr)^2} - \frac{1}{(kr)^3}\right]\sin\theta\,\hat{\theta}\right\} e^{i(kr-\omega t)}$$

using $\omega = ck$. Keeping only the leading terms in the $kr \gg 1$ limit we obtain the desired result.

(I-9, cont.)

b. $$\frac{dP}{d\Omega} = r^2 \frac{1}{2\mu_0} \mathrm{Re}\left(\vec{E} \times \vec{B}^*\right) \cdot \hat{r}$$

$$= \frac{\mu_0 \omega^4 |p_0|^2}{32 \pi^2 c} \sin^2\theta$$

c. $2^4 = 16$.

QE-Ph.D. January 2012 John Schroeder
(12·I·2012)

[I-10] Solution:

(a.) From momentum and energy conservation we may write

$$\vec{p} = \tilde{\vec{p}} + \vec{p}_e$$

$$\mathcal{E} + \mathcal{E}_e = \tilde{\mathcal{E}} + \tilde{\mathcal{E}}_e$$

where $\vec{p}, \tilde{\vec{p}}, \mathcal{E}, \tilde{\mathcal{E}}$ are the momenta and energies of the photon before and after the scattering, respectively, $\vec{p}_e$, $\tilde{\mathcal{E}}_e$ are the final momentum and energies of the electron, and $\mathcal{E}_e$ is its initial energy. We have for the electron

$$\tilde{\mathcal{E}}_e = \sqrt{p_e^2 c^2 + m^2 c^4}\,, \quad \mathcal{E}_e = mc^2$$

and for the photon

$$\mathcal{E} = pc\,, \quad \tilde{\mathcal{E}} = \tilde{p}c$$ we may now

rewrite the above equations in the form

$$\vec{p} - \tilde{\vec{p}} = \vec{p}_e$$

$$pc + mc^2 = \tilde{p}c + \sqrt{p_e^2 c^2 + m^2 c^4}$$

(b.) To solve these equations we may express the momentum of the recoil electron, p_e, in two ways:

[I-10] continued.

$$\vec{p}_e^{\,2} = (\vec{p} - \tilde{\vec{p}})^2$$

$$p_e^2 = (p - \tilde{p})^2 + 2mc(p - \tilde{p})$$ using the

respective equations from part (a.)

$$p\tilde{p}(1 - \cos\theta) = mc(p - \tilde{p})$$

and when $\theta = \pi/2$ $\cos\theta = 0$. We have

$$p\tilde{p} = mc(p - \tilde{p})$$ or we may

divide by $p\tilde{p}$ and obtain

$$1 - mc\left(\frac{1}{\tilde{p}} - \frac{1}{p}\right)$$ but $p = h/\lambda$

then the final result becomes

$$\tilde{\lambda} - \lambda = \frac{h}{mc} \,.$$

Physics Ph.D. Qualifying Exam
Part II January 2012

[II-1] **[10]**

$\psi(\vec{x},t)$ is a solution of the Schroedinger equation : $i\hbar\dfrac{\partial\psi}{\partial t}=-\dfrac{\hbar^2}{2m}\Delta\psi+V(\vec{x})\,\psi$.

Demonstrate that

$$\rho(\vec{x},t)=|\psi(\vec{x},t)|^2 \quad \text{and} \quad \vec{j}(\vec{x},t)=\frac{i\hbar}{2m}\left(\psi\vec{\nabla}\psi^*-\psi^*\vec{\nabla}\psi\right)$$

fulfill the continuity equation: $\dfrac{\partial\rho}{\partial t}+\vec{\nabla}\bullet\vec{j}=0$.

[II-2] **[10]**

A *plane* rotator with a moment of inertia I and an electric dipole moment $\boldsymbol{d}$ is placed in a uniform time-independent electric field ε (being in the plane of the rotation). Treating the interaction between the dipole and the electric field perturbatively, determine the first *non-vanishing* correction to the rotator's energy levels.

[II-3] **[10]**

An electron is in the spin up state with respect to the z-axis at time t=0, $S_z|\psi(0)\rangle=\frac{\hbar}{2}|\psi(0)\rangle$. It is subject to a constant uniform magnetic induction $\boldsymbol{B}$, oriented in an arbitrary direction. Thus, the Hamiltonian is given by $H=-\frac{e}{m}\boldsymbol{B}\cdot\boldsymbol{S}$. Find the probability $P(t)$ that the electron at a later time $t\neq 0$ will be in the spin up state with respect to the x-axis, $S_x|\chi\rangle=\frac{\hbar}{2}|\chi\rangle$. Your answer should be expressed in terms of the electron mass m, the charge e, the components and magnitude of $\boldsymbol{B}$, the time t, and of course, $\hbar$.
A formula that you will need to know is $e^{i\alpha\cdot\sigma}=\cos|\alpha|+i\hat{\alpha}\cdot\sigma\sin|\alpha|$.

[II-4]   [10]

Use the Born approximation to compute the total scattering cross section σ for particles of mass m from an attractive Gaussian potential,

$$V(r) = -V_o e^{-(r/a)^2}.$$

[II-5]   [10]

Show that in the nth state of the simple harmonic quantum oscillator

$$\langle x^2 \rangle = (\Delta x)^2 \text{ and } \langle p^2 \rangle = (\Delta p)^2.$$

[II-6]   [10]

Consider a charged one-dimensional harmonic oscillator with mass m, frequency ω_o, and charge q. Initially the oscillator is in its unperturbed ground state when there is no electric field present. At $t = 0$ a weak spatially uniform electric field $E = E_o e^{-\gamma t} \cos(\omega t)$ is imposed (the field is parallel to the direction of motion of the oscillator) with $\gamma < \omega_o$. Using *time-dependent perturbation theory*, find the transition probabilities to all excited states for $t = \infty$. For fixed ω_o and γ, what value of ω maximizes these transition probabilities? You may find the number representation of the harmonic oscillator with the annihilation and creation operators useful:

$$a = \sqrt{\frac{m\omega_o}{2\hbar}}\left(x + \frac{i}{m\omega_o}p\right),\quad a^+ = \sqrt{\frac{m\omega_o}{2\hbar}}\left(x - \frac{i}{m\omega_o}p\right).$$

[II-7]   [4,4,2]

The conditions for phase coexistence between a liquid and gas are

$$\mu_l = \mu_g, \quad p_l = p_g, \quad T_l = T_g .$$

(a) Considering that these equations hold for a p,T pair and for a nearby $p+dp$, $T+dT$ pair, derive a differential equation for dp/dT in terms of the partial derivatives of the chemical potentials with respect to p and T.

(b) Re-express this equation in terms of the entropies S_l, S_g and volumes V_l, V_g.

(c) Taking into account that the latent heat of vaporization is given by $L=T(S_g - S_l)/N$ where N is the number of molecules in the sample, obtain the *Clausius-Claperyron* equation.

[II-8]   [10]

Suppose a particular type of rubber band has the equation of state $l = \theta f / T$, where l the length per unit mass, f is the tension, T is the temperature, and θ is a constant. For this type of rubber band, compute $\left(\frac{\partial c_l}{\partial l}\right)_T$, where c_l is the constant length heat capacity per unit mass.

[II-9]   [2,8]

Consider a model system with single-particle energy levels $0, \varepsilon, 2\varepsilon, 3\varepsilon, 4\varepsilon, 5\varepsilon, \ldots$. The energy levels are *non-degenerate*. The system is completely isolated from the rest of the universe. There are $N = 3$ distinguishable (localized) particles in the system. The total energy of the system is $E = 5\varepsilon$.

(a) What is the entropy of the system subject to the above constraints?

(b) What is the expectation value of the number of particles in the lowest energy level?

[II-10]   [10]

Determine the pressure for a free, *non-relativistic electron gas* of volume V and N electrons in three dimensions, at zero temperature. Compare your result with the pressure of an ideal gas at the same temperature.

QE, January 2012
Part II – Solutions

$\boxed{\text{II}-\cancel{}1}$

$$\rho(\vec{x},t) = |\psi(\vec{x},t)|^2 = \psi^*(\vec{x},t)\psi(\vec{x},t)$$

$$\frac{\partial \rho}{\partial t} = \frac{\partial}{\partial t}(\psi^*\psi) = \dot{\psi}^*\psi + \psi^*\dot{\psi}$$

$$\vec{\nabla}\cdot\vec{j} = \frac{i\hbar}{2m}\vec{\nabla}(\psi\vec{\nabla}\psi^* - \psi^*\vec{\nabla}\psi)$$

$$\vec{\nabla}\cdot\vec{j} = \frac{i\hbar}{2m}(\vec{\nabla}\psi\,\vec{\nabla}\psi^* + \psi\Delta\psi^* - \vec{\nabla}\psi^*\vec{\nabla}\psi - \psi^*\Delta\psi)$$

$$\vec{\nabla}\cdot\vec{j} = \frac{i\hbar}{2m}(\psi\Delta\psi^* - \psi^*\Delta\psi)$$

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot\vec{j} = \dot{\psi}^*\psi + \psi^*\dot{\psi} + \frac{i\hbar}{2m}(\psi\Delta\psi^* - \psi^*\Delta\psi)$$

$$= \psi\left(\dot{\psi}^* + \frac{i\hbar}{2m}\Delta\psi^*\right) + \psi^*\left(\dot{\psi} - \frac{i\hbar}{2m}\Delta\psi\right)$$

from $i\hbar\dot{\psi} = -\frac{\hbar^2}{2m}\Delta\psi + V(\vec{x})\psi$

$$\Leftrightarrow \quad \dot{\psi} = -\frac{\hbar}{i2m}\Delta\psi + V(\vec{x})\psi \qquad -\frac{1}{i} = -\frac{i}{i\,i} = -\frac{i}{(-1)} = i$$

$$\dot{\psi} = \frac{i\hbar}{2m}\Delta\psi + \frac{V(\vec{x})\psi}{i\hbar}$$

$$\dot{\psi} - \frac{i\hbar}{2m}\Delta\psi = \frac{V(\vec{x})\psi}{i\hbar}$$

same for $\dot{\psi}^* + \frac{i\hbar}{2m}\Delta\psi^* = -\frac{V(\vec{x})}{i\hbar}\psi^*$

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot\vec{j} = \psi\frac{V(\vec{x})\psi}{i\hbar} + \psi^*\left(-\frac{V(\vec{x})}{i\hbar}\right)\psi^* = \underline{\underline{0}}$$

II-2. Solution

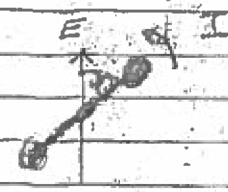
(II-2)

(use $\bar{\mathcal{E}}$ for electric field to avoid confusion.)

-1) Plane Rotator

$$H_0\psi = E^{(0)}\psi$$

$$-\frac{\hbar^2}{2I}\frac{d^2\psi}{d\varphi^2} = E^{0}\psi$$



unperturbed Hamiltonian $H_0 = -\frac{\hbar^2}{2I}\frac{d^2}{d\varphi^2}$

perturbation: $H^1 = -\overset{\text{electric field}}{E}\cdot d\cos\varphi$ $\qquad E = |\bar{E}|,\ d = |\bar{d}|$

$$(H_0 + H^1)\psi = E\psi$$

unperturbed eigenfunctions & eigenvalues:

$$\psi_m^{(0)}(\varphi) = \frac{1}{\sqrt{2\pi}} e^{im\varphi} \qquad m = 0, \pm 1, \pm 2, \ldots$$

$$E_m^{(0)} = \frac{\hbar^2 m^2}{2I}$$

Despite the 2-fold degeneracy, non-degenerate perturbation theory can be applied since, $\langle \psi_{-m}^{(0)} | H^1 | \psi_m^{(0)} \rangle = 0$ for all m

i.e, the $-m$ state does not give any contribution to the corrections of the $+m$ state.

$$H'_{mm'} = \int_0^{2\pi} \psi_m^{(0)*} H^1 \psi_{m'}^{(0)}\, d\varphi = -\frac{Ed}{2\pi}\int_0^{2\pi} e^{i(m'-m)\varphi}\cos\varphi\, d\varphi$$

$$= -\frac{Ed}{2\pi}\,\frac{1}{2}\left\{\int_0^{2\pi} e^{i(m'-m+1)\varphi}\, d\varphi + \int_0^{2\pi} e^{i(m'-m-1)\varphi}\, d\varphi\right\} =$$

$$= \frac{-Ed}{2}\left(\delta_{m',m-1} + \delta_{m',m+1}\right) \quad ; \text{ only } m' = m \pm 1 \text{ nonzero elements}$$

$$\left(H'_{m,-m} = 0 \quad \forall m \right)$$

(II-2 cont.)

$E_m^{(1)} = H'_{mm} = 0$, so one must go to 2^{nd} order (formally non-degenerate) perturbation theory

$$E_m^{(2)} = \sum_{j \neq m} \frac{|H'_{mj}|^2}{E_m^{(0)} - E_j^{(0)}} = \frac{|H'_{m\,m-1}|^2}{E_m^{(0)} - E_{m-1}^{(0)}} + \frac{|H'_{m,m+1}|^2}{E_m^{(0)} - E_{m+1}^{(0)}} =$$

$$= \frac{E^2 d^2}{4} \left\{ \frac{1}{\frac{\hbar^2 m^2}{2I} - \frac{\hbar^2 (m-1)^2}{2I}} + \frac{1}{\frac{\hbar^2 m^2}{2I} - \frac{\hbar^2 (m+1)^2}{2I}} \right\}$$

$$= \frac{I E^2 d^2}{2\hbar^2} \left\{ \frac{1}{m^2 - (m-1)^2} + \frac{1}{m^2 - (m+1)^2} \right\} = \frac{I E^2 d^2}{2\hbar^2} \left\{ \frac{1}{2m-1} + \frac{1}{-2m-1} \right\}$$

$$= \frac{I E^2 d^2}{2\hbar^2} \left\{ \frac{1}{2m-1} - \frac{1}{2m+1} \right\} = \frac{I E^2 d^2}{2\hbar^2} \cdot \frac{2m+1 - (2m-1)}{(2m-1)(2m+1)}$$

$$= \frac{I E^2 d^2}{\hbar^2} \frac{1}{4m^2 - 1}$$

Thus,

$$\boxed{E_m \simeq \frac{\hbar^2 m^2}{2I} + \frac{I E^2 d^2}{\hbar^2} \frac{1}{4m^2 - 1}}$$

$m = 0, \pm 1, \pm 2, \ldots$

Problem II-3

$$H = -\frac{e\hbar}{2m}\vec{B}\cdot\vec{\sigma} = -\frac{e}{m}\vec{B}\cdot\vec{S}$$

$$e^{-iHt/\hbar} = \exp\left[\frac{iet}{2m}\vec{B}\cdot\vec{\sigma}\right]$$

$$= \cos\left(\frac{e}{2m}|\vec{B}|t\right) + i\,\hat{B}\cdot\vec{\sigma}\sin\left(\frac{e}{2m}|\vec{B}|t\right)$$

$$S_x|\psi\rangle = \frac{\hbar}{2}|\psi\rangle$$

$$\Rightarrow |\psi\rangle = \frac{1}{\sqrt{2}}\left(|+\rangle + |-\rangle\right)$$

$$\langle\psi|e^{-iHt/\hbar}|+\rangle$$

$$= \frac{1}{\sqrt{2}}\langle +|e^{-i\frac{Ht}{\hbar}}|+\rangle + \frac{1}{\sqrt{2}}\langle -|e^{-i\frac{Ht}{\hbar}}|+\rangle$$

$$= \frac{1}{\sqrt{2}}\cos\left(\frac{e}{2m}|\vec{B}|t\right) + \frac{1}{\sqrt{2}}i\sin\left(\frac{e}{2m}|\vec{B}|t\right)\left(\hat{B}_1 + i\hat{B}_2\right)$$
$$+ \frac{i}{\sqrt{2}}\hat{B}_3\sin\left(\frac{e|\vec{B}|}{2m}t\right)$$

$$P(t) = \frac{1}{2}\left(\cos(\omega t) - \hat{B}_2\sin(\omega t)\right)^2$$
$$+ \frac{1}{2}\left(\hat{B}_3 + \hat{B}_1\right)^2\sin^2(\omega t)$$

$$\omega = \frac{e|\vec{B}|}{2m}, \quad \hat{B}_i = \frac{B_i}{|\vec{B}|}$$

II-4

Use the Born approximation to comptite the total scattering cross section σ for particles of mass m from an attractive Gaussian potential,

$$V(r) = -V_0 e^{-(r/a)^2}$$

Solution:

The scattering amplitude $f(\theta)$ is proportional to the 3D Fourier transform of the potential. Recall that the Fourier transform of a Gaussian is another Gaussian,

$$f(\theta,\phi) = -\frac{m}{2\pi\hbar^2}\int e^{-iq\cdot r'}V(r')d^3r' \tag{0.3}$$

$$= \frac{mV_0}{2\pi\hbar^2}\int e^{-iq\cdot r'}e^{-(r/a)^2}d^3r' \tag{0.4}$$

$$= \frac{\sqrt{\pi}V_0 ma^3}{2\hbar^2}e^{-q^2a^2/4} \tag{0.5}$$

The differential cross section is

$$\frac{d\sigma}{d\Omega} = |f(\theta)|^2 = \frac{\pi m^2 V_0^2 a^6}{4\hbar^4}e^{-q^2a^2/2}$$

The total cross section is the integral of $\frac{d\sigma}{d\Omega}$ over the solid angles. Recall that

$$q^2 = 4k^2 sin^2(\theta/2)$$

Then

$$\sigma = \int \frac{d\sigma}{d\Omega}d\Omega \tag{0.6}$$

$$= \frac{\pi m^2 V_0^2 a^6}{4\hbar^4}\int_0^{2\pi} d\phi \int_0^{\pi} e^{-2k^2a^2 sin^2(\theta/2)} sin\theta d\theta \tag{0.7}$$

$$= 2\pi\frac{\pi m^2 V_0^2 a^6}{4\hbar^4}\int_0^{\pi} e^{-2k^2a^2 sin^2(\theta/2)} sin\theta d\theta \tag{0.8}$$

$$= \frac{\pi^2 m^2 V_0^2 a^6}{2\hbar^4}\int_0^{2k^2a^2} \frac{1}{k^2a^2}e^{-x}dx \tag{0.9}$$

$$\sigma = \frac{\pi^2 m^2 V_0^2 a^4}{2\hbar^4 k^2}(1 - e^{-2k^2a^2}) \tag{0.10}$$

$\boxed{\text{II-5}}$ Wavepacket/Uncertainty Principle/Partial Waves

Show that in the nth state of the harmonic oscillator:
$\langle x^2\rangle = (\Delta x)^2$ and $\langle p^2\rangle = (\Delta p)^2$.

$(\Delta x)^2 = \langle (x - \langle x\rangle)^2\rangle = \langle x^2\rangle$ because $\langle x\rangle = 0$ in the nth state.

$(\Delta p)^2 = \langle (p - \langle p\rangle)^2\rangle = \langle p^2\rangle$ $\langle p\rangle = 0$ in nth state

$\langle x\rangle = \langle n|\hat{x}|n\rangle$

$= \frac{1}{\sqrt{2}\beta} \langle n|\hat{a} + \hat{a}^*|n\rangle$

$= \frac{1}{\sqrt{2}\beta} \{ n^{1/2} \langle n|n-1\rangle + (n+1)^{1/2} \langle n|n+1\rangle \}$

$= 0$ because $\langle n|n-1\rangle = 0$
$\langle n|n+1\rangle = 0$

$\langle p\rangle = \langle n|\hat{p}|n\rangle = \frac{m\omega_0}{\sqrt{2}\, i\beta} \langle n|\hat{a} - \hat{a}^*|n\rangle$

$= \frac{m\omega_0}{\sqrt{2}\, i\beta} (n^{1/2}\langle n|n-1\rangle - (n+1)^{1/2}\langle n|n+1\rangle$

$= 0$

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$\boxed{\text{II-6}}$ Time-dependent Perturbation

$H = H_0 + V(t)$ $\qquad V(t) = -qE_0 x\, e^{-\gamma t} \cos(\omega t)$ $(t>0)$

(charged) harmonic oscillator $\quad H_0 = \frac{p^2}{2m} + \frac{m\omega_0^2}{2} x^2$

$$\left.\begin{aligned} a &= \sqrt{\frac{m\omega_0}{2\hbar}} \left(x + \frac{i}{m\omega_0} p\right) \\ a^+ &= \sqrt{\frac{m\omega_0}{2\hbar}} \left(x - \frac{i}{m\omega_0} p\right) \end{aligned}\right\} \Rightarrow \quad x = \sqrt{\frac{\hbar}{2m\omega_0}} (a + a^+)$$

$$H_0 |n\rangle = \hbar\omega_0 (n + 1/2) |n\rangle \qquad n = 0, 1, 2, \ldots$$

$$E_n = \hbar\omega_0 (n + 1/2) \qquad \omega_{\ell k} \equiv \frac{1}{\hbar}(E_\ell - E_k) = \omega_0 (\ell - k)$$

$t = 0$: oscillator is in G.S. $|0\rangle$

Transition probabilities for $t = \infty$:

$$W(0 \to \ell) = \frac{1}{\hbar^2} \left| \int_0^\infty \langle \ell | V(t) | 0 \rangle e^{i\omega_{\ell 0} t} dt \right|^2$$

$$= \frac{q^2 E_0^2}{\hbar^2} |\langle \ell | x | 0 \rangle|^2 \left| \int_0^\infty e^{i\omega_{\ell 0} t} e^{-\gamma t} \cos(\omega t)\, dt \right|^2$$

$$= \frac{q^2 E_0^2}{\hbar^2} \frac{\hbar}{2m\omega_0} |\langle \ell | 1 \rangle|^2 \left| \int_0^\infty e^{i\omega_0 \ell t} e^{-\gamma t} \frac{e^{i\omega t} + e^{-i\omega t}}{2i} dt \right|^2$$

$$= \delta_{\ell 1} \frac{q^2 E_0^2}{4\hbar m\omega_0} \frac{1}{2} \left| \int_0^\infty \left\{ e^{[i(\omega_0 + \omega) - \gamma] t} + e^{[i(\omega_0 - \omega) - \gamma] t} \right\} dt \right|^2 =$$

1) Kronecker-delta

$\boxed{\text{II} - 6}$ (Cont'd)

$$= \delta_{e1} \frac{q^2 E_0^2}{4\hbar m\omega_0 2} \left| \frac{1}{\gamma - i(\omega_0+\omega)} + \frac{1}{\gamma - i(\omega_0-\omega)} \right|^2$$

$$= \delta_{e1} \frac{q^2 E_0^2}{4\hbar m\omega_0 2} \left| \frac{\gamma - i(\omega_0-\omega) + \gamma - i(\omega_0+\omega)}{[\gamma - i(\omega_0+\omega)][\gamma - i(\omega_0-\omega)]} \right|^2 =$$

$$= \delta_{e1} \frac{q^2 E_0^2}{4\hbar m\omega_0} \left| \frac{2(\gamma - i\omega_0)}{\gamma^2 - i\gamma(\omega_0-\omega) - i\gamma(\omega_0+\omega) - (\omega_0-\omega)(\omega_0+\omega)} \right|^2$$

$$= \delta_{e1} \frac{q^2 E_0^2}{4\hbar m\omega_0 2} \left| \frac{2(\gamma - i\omega_0)}{\gamma^2 - 2i\gamma\omega_0 + (\omega^2 - \omega_0^2)} \right|^2$$

$$= \delta_{e1} \frac{q^2 E_0^2}{\cancel{4}\hbar m\omega_0 \cancel{2}} \cancel{4} \frac{\gamma^2 + \omega_0^2}{[\gamma^2 + \omega^2 - \omega_0^2]^2 + 4\gamma^2\omega_0^2}$$

for $\gamma < \omega_0$, $W(0 \to e)$ is maximum if $\omega = \sqrt{\omega_0^2 - \gamma^2}$.

transition is only possible to the first excited state $|1\rangle$, and the tr. prob. is maximum if $\omega = \sqrt{\omega_0^2 - \gamma^2}$.

Problem II-7

a. $\mu_\ell(p+dp, T+dT) = \mu_g(p+dp, T+dT)$

$$\mu_\ell(p,T) + \left(\frac{\partial \mu_\ell}{\partial p}\right)_T dp + \left(\frac{\partial \mu_\ell}{\partial T}\right)_p dT$$

$$= \mu_g(p,T) + \left(\frac{\partial \mu_g}{\partial p}\right)_T dp + \left(\frac{\partial \mu_g}{\partial T}\right)_p dT$$

$$\frac{dp}{dT} = \frac{\left(\frac{\partial \mu_\ell}{\partial T}\right)_p - \left(\frac{\partial \mu_g}{\partial T}\right)_p}{\left(\frac{\partial \mu_g}{\partial p}\right)_T - \left(\frac{\partial \mu_\ell}{\partial p}\right)_T}$$

b. $G = N\mu(p,T)$

$$\left(\frac{\partial G}{\partial p}\right)_{N,T} = V, \quad \left(\frac{\partial G}{\partial T}\right)_{N,p} = -S$$

$$V_\ell = N\left(\frac{\partial \mu_\ell}{\partial p}\right)_T, \quad V_g = N\left(\frac{\partial \mu_g}{\partial p}\right)_T$$

$$S_\ell = -N\left(\frac{\partial \mu_\ell}{\partial T}\right)_p, \quad S_g = -N\left(\frac{\partial \mu_g}{\partial T}\right)_p$$

(II-7, cont)

$$\frac{dp}{dT} = \frac{S_g - S_\ell}{V_g - V_\ell}$$

c. $$\frac{dp}{dT} = \frac{NL}{T} \frac{1}{V_g - V_\ell} = \frac{L}{T \Delta v}$$

$$\Delta v = \frac{V_g - V_\ell}{N}$$

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[II-8] Solution

From the definition of the heat capacity

$$\left(\frac{\partial c_\ell}{\partial \ell}\right)_T = \left(\frac{\partial}{\partial \ell} T \left(\frac{\partial S}{\partial T}\right)_\ell\right)_T = T \left(\frac{\partial}{\partial T}\left(\frac{\partial S}{\partial \ell}\right)_T\right)_\ell .$$

Further, since $dE = T\,dS + f\,d\ell + \mu\,dn$

we have: $d(E-TS) = -S\,dT + f\,d\ell + \mu\,dn$

and hence the Maxwell relation gives

$$\left(\frac{\partial S}{\partial \ell}\right)_T = -\left(\frac{\partial f}{\partial T}\right)_\ell$$

and now

$$\left(\frac{\partial c_\ell}{\partial \ell}\right)_T = -T\left(\frac{\partial^2 f}{\partial T^2}\right)_\ell = -T\,\frac{\partial^2}{\partial T^2}\left(\frac{\ell T}{\theta}\right) = 0 .$$

II-9

(N_{jk}) $j \backslash k$	1.	2.	3.	4.	5.
0	2	1	1		
ε		1		2	1
2ε			1		2
3ε			1	1	
4ε		1			
5ε	1				
$\vdots$					

"macrostates" (k.)

$$N = \sum_j N_j = 3$$
$$E = \sum_j N_j \varepsilon_j = 5\varepsilon$$

- non-degenerate levels
- distinguishable particles

$$w_k = w(N_0, N_1, \dots) = \frac{N!}{\prod_j N_j!} \quad \left(\text{with } \sum N_j = 3,\ \sum N_j \varepsilon_j = 5\varepsilon\right)$$

# of microstates in macrostate "k"
$k = 1, 2, 3, 4, 5$

$$w_1 = \frac{3!}{2!1!} = 3$$
$$w_2 = \frac{3!}{1!1!1!} = 6$$
$$w_3 = \frac{3!}{1!1!1!} = 6$$
$$w_4 = \frac{3!}{2!1!} = 3$$
$$w_5 = \frac{3!}{1!2!} = 3$$

$$P_k = \frac{w_k}{\Omega}$$ probability of the system being in macrostate "k"

$$\Omega = \sum_k w_k$$ total # of microstates

a) $\Omega = \sum_k w_k = 21$ $\quad S(N=3, E=5\varepsilon) = k_B \ln \Omega = \boxed{k_B \ln(21)}$

b) $$\overline{N_j} = \sum_k P_k N_{jk} = \sum_k \frac{w_k}{\Omega} N_{jk} = \frac{1}{\Omega} \sum_k w_k N_{jk}$$

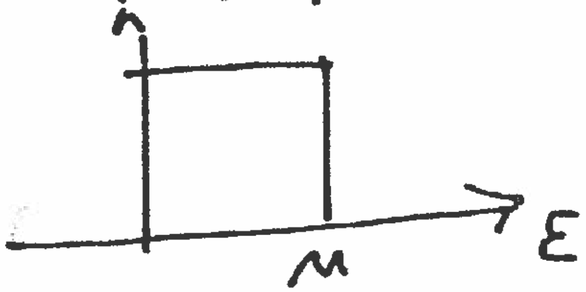
$\boxed{j=0:}$ $\overline{N_0} = \frac{1}{21}\{3\cdot 2 + 6\cdot 1 + 6\cdot 1\} = \boxed{\frac{18}{21}}$

Problem $\boxed{II-10}$

Electron → Fermi-Dirac

$$n(p) = \frac{1}{e^{\beta(\varepsilon-\mu)}+1} \qquad \beta = \frac{1}{kT}$$

at $T=0$ it becomes a step function



$\mu = \varepsilon_F$

Thermodynamics

$$N = \underbrace{(2s+1)}_{2} \frac{1}{h^3} \underbrace{\int d^3p}_{4\pi\int p^2 dp} \underbrace{\int d^3x}_{V}\, n(p)$$

$$= \frac{8\pi V}{h^3}\int p^2 n(p)\, dp$$

$$\equiv \frac{8\pi V}{h^3}\int g(\varepsilon)\, n(\varepsilon)\, d\varepsilon$$

$g(\varepsilon)$: $\quad \varepsilon = \frac{p^2}{2m} \qquad d\varepsilon = \frac{p}{m}dp$

$$g(\varepsilon) = m\sqrt{2m\varepsilon}$$

(II-10, cont)

$$N = \frac{8\pi V m\sqrt{2m}}{h^3} \int_0^{\varepsilon_F} \sqrt{\varepsilon}\, d\varepsilon$$

$$= \frac{2}{3} \cdot \frac{8\pi}{h^3} \cdot V \cdot \sqrt{2m}\, \varepsilon_F^{3/2}$$

This gives

$$\varepsilon_F = \left(\frac{3}{8\pi}\right)^{2/3} \frac{h^2}{2m} \left(\frac{N}{V}\right)^{2/3}$$

Now we calculate the energy E:

$$E = 2 \int g(\varepsilon) n(\varepsilon)\, \varepsilon\, d\varepsilon$$

at $T=0$

$$E = 2 \int_0^{\varepsilon_F} g(\varepsilon)\, \varepsilon\, d\varepsilon$$

$$= \frac{4\pi V}{h^3} (2m)^{3/2} \int_0^{\varepsilon_F} \varepsilon^{3/2} d\varepsilon$$

$$= \frac{8\pi V}{5 h^3} (2m)^{3/2} \varepsilon_F^{5/2}$$

$$E = \frac{3}{10} \left(\frac{3}{8\pi}\right)^{2/3} \frac{h^2}{2m} \left(\frac{N}{V}\right)^{2/3} N$$

$$p = -\frac{\partial E}{\partial V} = \frac{1}{5} \left(\frac{3}{8\pi}\right)^{2/3} \frac{h^2}{2m} \left(\frac{N}{V}\right)^{5/3} \neq 0$$

at $T=0$ $p=0$ for an ideal gas.