Physics PhD Qualifying Examination
Part 1 – Wednesday, January 15, 2014

Name:

(please print)

Identification Number:

**STUDENT:** Designate the problem numbers that you are handing in for grading in the appropriate left hand boxes below. Initial the right hand box.

**PROCTOR:** Check off the right hand boxes corresponding to the problems received from each student. Initial in the right hand box.

<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td></td>
<td></td>
</tr>
<tr>
<td>9</td>
<td></td>
<td></td>
</tr>
<tr>
<td>10</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Student's initials**

**# problems handed in:**

**Proctor's initials**

**INSTRUCTIONS FOR SUBMITTING ANSWER SHEETS**

1. DO NOT PUT YOUR NAME ON ANY ANSWER SHEET. EXAMS WILL BE COLLATED AND GRADED BY THE ID NUMBER ABOVE.
2. Use at least one separate preprinted answer sheet for each problem. Write on only one side of each answer sheet.
3. Write your **identification number** listed above, in the appropriate box on each preprinted answer sheet.
4. Write the **problem number** in the appropriate box of each preprinted answer sheet. If you use more than one page for an answer, then number the answer sheets with both problem number and page (e.g. Problem 9 – Page 1 of 3).
5. Staple together all the pages pertaining to a given problem. Use a paper clip to group together all problems that you are handing in.
6. A passing distribution for the individual components will normally include at least three passed problems (from problems 1-5) for Mechanics and three problems (from problems 6-10) for Electricity and Magnetism.
7. **YOU MUST SHOW ALL YOUR WORK.**
A particle of mass \( m \) is bound by a linear potential \( U(r) = kr \).

(a) For what energy and angular momentum will the orbit be a circle of radius \( r \) about the origin?
(b) What is the frequency of this circular motion?
(c) If the particle is slightly disturbed from this circular motion, what will be the frequency of small oscillations?

A particle of mass \( m \) is constrained to move (without friction) on a circular wire with radius \( a \), which is rotating with a fixed angular velocity \( \Omega \) about its vertical diameter in the presence of uniform gravity \( g \).

(a) Find a convenient generalized coordinate and write down the Lagrangian of this system.
(b) Obtain the equation of motion.
(c) Find all possible equilibrium positions of the particle along the circular wire.
(d) Discuss the stability of the above equilibrium points depending on the value of the angular velocity \( \Omega \) and obtain the frequency of small oscillations about the respective stable equilibrium position.
Consider the coupled mass-spring system sketched below. The masses can only move horizontally. The springs are relaxed at equilibrium. Obtain the complete solution of the problem for the following set of initial conditions:

\[ x_1(0) = D, \quad x_2(0) = 0; \quad \dot{x}_1(0) = 0, \quad \dot{x}_2(0) = 0. \]

Note that simply finding the normal frequencies and normal modes is not sufficient for obtaining a passing score for this problem. You have to obtain \( x_1(t) \) and \( x_2(t) \) for the given initial conditions. (The displacements \( x_1(t) \) and \( x_2(t) \) are measured from the respective equilibrium positions of the masses.)

---

Consider earth with a total mass \( M \) and radius \( R \). Also, assume that earth's mass is distributed uniformly.

(a) Argue that one may use the equivalent of "Gauss law" to determine gravity inside and outside of the earth's surface. State the "Gauss law".

(b) Derive functional form of gravity \( g \) as a function of radial distance \( r \) inside and outside the earth's surface. Sketch \( g \) vs \( r \). (should mark relevant quantities and functional dependence.)

(c) Derive gravitational potential \( U(r) \) as a function of \( r \) inside and outside the earth's surface. Sketch \( U \) vs \( r \). (should mark relevant quantities and functional dependence.)
A stick of lengths \( l \) is fixed at an angle \( \theta \) from its \( x_1 \)-axis in its own rest system \( \mathcal{K} \). What is the lengths and orientation of the stick as measured by an observer moving along \( x_1 \) with relativistic speed \( v \)?
Suppose that the region $z > 0$ in three-dimensional space is filled with a linear dielectric material characterized by a dielectric constant $\epsilon_1$, while the region $z < 0$ has a dielectric material $\epsilon_2$. Fix a charge $-q$ at $(x, y, z) = (0, 0, a)$ and a charge $+q$ at $(0, 0, -a)$. What is the force one must exert on the negative charge to keep it at rest?

Consider a conducting rod of mass $m$ on a tilted pair of rails in the presence of gravity $g$ and a constant and homogeneous vertical magnetic field $B$ (see sketch below). The angle of inclination is $\alpha$. The distance between the rails is $l$. The rails are connected with an inductance $L$. The resistance of the rod and the rails and the friction between the rod and the rails are negligible. Initially the rod is at rest and there is no current in the loop ($v(0) = 0$, $i(0) = 0$). Describe the motion of the rod, i.e., obtain $v(t)$. 

\[ \text{Sketch of the conducting rod on tilted rails with a magnetic field and gravity.} \]
Find the magnetic field at point O (the origin of z-axis) on the axis of a tightly wound solenoid. The solenoid consists of $N$ turns per unit length wrapped around a cylindrical tube of radius $R$ and carrying current $I$. The top and bottom of the solenoid are at a distance $z_1$ and $z_2$ from point O, respectively. A schematic of the solenoid is shown below. Note that you must carry out the integral.

(Hint: as the first step, you may start by first deriving $B$-field expression for a single current-loop.)
An electromagnetic wave $E = E_0 \exp(i(kz - \omega t))$ is incident on an atom of polarizability $\alpha$ located at the origin $(0, 0, 0)$ of a cartesian coordinate system.

(a) Find the electric and magnetic fields of the wave radiated by the atom at large distance $D$ on the $y$-axis and on the $x$-axis.

(b) Find the total time-averaged power radiated by the polarized atom.

Treat the atom as a dipole.

Two large (non-conducting) parallel plates separated by a distance $d$ and oriented as shown in the figure below, move together along the $x$-axis with a velocity $v$. The upper and lower plates have uniform charge density of $+\sigma$ and $-\sigma$, respectively, in the rest-frame of the plates.

(a) Find the magnitude and direction of the electric and magnetic fields between the plates.

(b) Now, if the parallel plates are tilted such that their surface normal is oriented along the direction of $\hat{y} + \hat{z}$. Find the magnitude and direction of the electric and magnetic fields between the plates.
A particle of mass $m$ is bound by a linear potential $U = kr$.

(a) For what energy and angular momentum will the orbit be a circle of radius $r$ about the origin?

(b) What is the frequency of this circular motion?

(c) If the particle is slightly disturbed from this circular motion, what will be the frequency of small oscillations?

**Solution:**

The force acting on the particle is $F = -\frac{\partial U}{\partial r} \hat{r} = -kr \hat{r}$.

(a) If the particle moves in a circle of radius $r$, $m \omega^2 r = k$, so $\omega^2 = k/m r$. The energy of the particle is then $E = kr + \frac{mv^2}{2} = kr + \frac{m \omega^2 r^2}{2} = \frac{3kr}{2}$, and its angular momentum about the origin is $L = m \omega r^2 = mr^2 \sqrt{k/mr} = \sqrt{mk}r^3$.

(b) The angular frequency of circular motion is $\omega = \frac{k}{\sqrt{mk}}$.

(c) The effective radial potential is $U_{eff} = kr + \frac{l^2}{2mr^2}$. The radius $r_0$ of stationary circular motion is given by $\left(\frac{\partial U_{eff}}{\partial r}\right)_{r=r_0} = k - \frac{k^2}{m r_0^2} = 0$, yielding $r_0 = \left(\frac{k^2}{mk}\right)^{1/2}$.

The frequency of small oscillations about $r_0$ is given by $\omega_s = \sqrt{\frac{k_{eff}}{m}}$, with $k_{eff} = \left(\frac{\partial^2 U_{eff}}{\partial r^2}\right)_{r=r_0} = 3l^2/m r_0^3 = 3kr_0^2/l^2$, $\omega_0 = \sqrt{\frac{3kr_0^2}{ml^2}} = \sqrt{\frac{3k}{ml^2}} = \sqrt{\frac{3k}{ml^2}}$, where $\omega_0$ is the frequency of circular motion.
a) \[ \ell = \frac{1}{2} \left( \alpha^2 \sin^2 \varpi + \alpha^2 \right) + mg \cos \varpi \]

\( \varpi \) is the only generalized coordinate.

b) \[ \frac{d^2 \varpi}{dt^2} = \frac{d\ell}{d\varpi} \]

\[ m \ddot{\varpi} = 2m \dot{\varpi} \sin \varpi \cos \varpi - mg \sin \varpi \]

\[ \ddot{\varpi} = \left( \frac{\beta}{\alpha} \right) \sin \varpi - \frac{g}{a} \sin \varpi \]

C) stationary angle \( \varpi \):

\( \varpi = \text{const} \). \( \ddot{\varpi} = 0, \dot{\varpi} = 0 \)

\[ \varpi = \left( \frac{\beta}{\alpha} \cos \varpi - \frac{g}{a} \right) \sin \varpi \]

\[ \varpi = \left( \frac{\beta}{\alpha} \cos \varpi - \beta_0 \right) \sin \varpi \]

i) \( \sin \varpi = 0 \Rightarrow \]

\[ \varpi = 0 \]

\[ \varpi_1 = \frac{g}{\beta} \]

ii) \( \beta_0 \cos \varpi - \beta_0 = 0 \Rightarrow \]

\[ \cos \varpi_3 = \frac{\beta_0}{\beta} \]

(\( \beta_0 > \beta_0 \))

\[ \varpi_3 = \cos^{-1} \left( \frac{\beta_0}{\beta} \right) \]
d) expand equation of motion about respective equilibrium positions up to linear order in \( \Theta - \Theta^* \)

\[
\ddot{\Theta} = \frac{1}{2} \Omega_0^2 \sin \Theta \cos \Theta - \Omega_0^2 \sin \Theta = \\
\approx \frac{1}{2} \Omega_0^2 \sin \Theta^* \cos \Theta^* - \Omega_0^2 \sin \Theta^* + \\
+ \frac{\Omega_0^2}{2} \left( \cos \Theta^* - \sin \Theta^* \right) - \Omega_0^2 \cos \Theta^* \left( \Theta - \Theta^* \right) + \ldots
\]

stability: \( \omega^2 > 0 \)

\[
\text{for } \Theta^* = 0: \quad \omega = \Omega_0 \sqrt{2} \quad \text{is eq.}
\]

\[
\text{for } \Theta^* = 0: \quad \omega = \sqrt{2} \Omega_0 \quad \text{if } \Omega_0 < \Omega_0^* \quad \text{i.e. } \Theta^* = 0 \text{ stable if } \Omega_0 < \sqrt{\frac{2}{\alpha}} \quad \text{and } \omega = \Omega_0 \sqrt{2}\]

\[
\text{for } \Theta^* = \pi, \Theta^* = \pi: \quad \omega^2 + \omega_0^2 \quad \text{is always } > 0
\]

\[
\text{never stable}
\]

\[
\text{for } \Theta^* = \Theta_0^* = \cos \left( \frac{5 \Omega_0^2}{15 \Omega_0^2} \right): \\
\Omega_0^2 (2 \frac{\Omega_0^2}{15 \Omega_0^2} - 1) - \Omega_0^2 \frac{5 \Omega_0^2}{15 \Omega_0^2} \Theta_0^* = \frac{5 \Omega_0^4}{15 \Omega_0^2} - 1 < 0 \quad \text{if } \Omega_0 > \Omega_0^*
\]

\[
\text{i.e. } \Theta^* \text{ is stable if } \Omega_0 > \Omega_0^* \text{ and } \omega = \Omega_0 \sqrt{\Omega_0^2 - \frac{5 \Omega_0^4}{15 \Omega_0^2}}
\]

Summing up "phase diagram":

\[
\Omega_0 < \sqrt{\frac{2}{\alpha}}: \quad \Theta^* = 0 \text{ is stable and } \omega = \Omega_0 \sqrt{\frac{1 - \frac{\Omega_0^2}{\Omega_0^2}}}{\Omega_0^2}
\]

\[
- \Omega_0 > \sqrt{\frac{2}{\alpha}}: \quad \Theta^* = \cos \left( \frac{\Omega_0^2}{15 \Omega_0^2} \right) \text{ is stable and } \omega = \Omega_0 \sqrt{\frac{1 - \frac{5 \Omega_0^2}{\Omega_0^2}}{\Omega_0^2}}
\]
I-3) continual.

Complete solution of the problem with given initial values:
(non-degenerate eigenvalues)

\[
\begin{align*}
\mathbf{x}(t) &= \begin{bmatrix} A_1 \cos(\omega_1 t) + B_1 \sin(\omega_1 t) \\ A_2 \cos(\omega_2 t) + B_2 \sin(\omega_2 t) \end{bmatrix} e^{\mathbf{A}_1 t} + \begin{bmatrix} A_1 \cos(\omega_1 t) + B_1 \sin(\omega_1 t) \\ A_2 \cos(\omega_2 t) + B_2 \sin(\omega_2 t) \end{bmatrix} e^{\mathbf{A}_2 t} \\
\mathbf{x}(0) &= \begin{bmatrix} A_1 \cos(\omega_1 t) + B_1 \sin(\omega_1 t) \\ A_2 \cos(\omega_2 t) + B_2 \sin(\omega_2 t) \end{bmatrix}
\end{align*}
\]

\[\begin{align*}
(1) \quad &\begin{cases} 
\dot{x}_1(t) = A_1 \cos(\omega_1 t) + B_1 \sin(\omega_1 t) + A_2 \cos(\omega_2 t) + B_2 \sin(\omega_2 t) \\
\dot{x}_2(t) = -A_1 \cos(\omega_1 t) - B_1 \sin(\omega_1 t) + A_2 \cos(\omega_2 t) + B_2 \sin(\omega_2 t)
\end{cases} \\
(2) \quad &\begin{cases} 
\dot{x}_1(t) = -\omega_1 A_1 \sin(\omega_1 t) + \omega_1 B_1 \cos(\omega_1 t) - \omega_2 A_2 \sin(\omega_2 t) + \omega_2 B_2 \cos(\omega_2 t) \\
\dot{x}_2(t) = \omega_1 A_1 \sin(\omega_1 t) - \omega_1 B_1 \cos(\omega_1 t) - \omega_2 A_2 \sin(\omega_2 t) + \omega_2 B_2 \cos(\omega_2 t)
\end{cases}
\end{align*}\]

\[\begin{align*}
\mathbf{x}(0) &= \begin{bmatrix} D \\ 0 \end{bmatrix} \quad \mathbf{x}_2(0) = \begin{bmatrix} 0 \\ \mathbf{D} \end{bmatrix}
\end{align*}\]

From (1):

\[D = A_1 + A_2 \quad D = -A_1 + A_2 \quad \Rightarrow \quad A_1 = A_2 = \frac{D}{2}\]

From (2):

\[\begin{align*}
\omega_1 B_1 + \omega_2 B_2 &= 0 \\
-\omega_1 B_1 + \omega_2 B_2 &= 0
\end{align*}\]

\[\Rightarrow \quad B_1 = B_2 = 0\]

Full solution:

\[\begin{align*}
x_1(t) &= \frac{D}{2} \left( \cos(\omega_1 t) + \cos(\omega_2 t) \right) = D \cos\left(\frac{\omega_1 + \omega_2}{2} t\right) \cdot \cos\left(\frac{\omega_1 - \omega_2}{2} t\right) \\
x_2(t) &= \frac{D}{2} \left( -\cos(\omega_1 t) + \cos(\omega_2 t) \right) = D \sin\left(\frac{\omega_1 + \omega_2}{2} t\right) \cdot \sin\left(\frac{\omega_1 - \omega_2}{2} t\right)
\end{align*}\]
\[ \omega_1 = \sqrt{\frac{1}{m}} , \quad \omega_2 = \sqrt{\frac{E}{m}} \]

\[ x_1(t) = D \cos \left( \sqrt{\frac{E}{m}} \frac{\sqrt{3} + 1}{2} t \right) \cos \left( \sqrt{\frac{E}{m}} \frac{\sqrt{3} - 1}{2} t \right) \]

\[ x_2(t) = D \sin \left( \sqrt{\frac{E}{m}} \frac{\sqrt{3} + 1}{2} t \right) \sin \left( \sqrt{\frac{E}{m}} \frac{\sqrt{3} - 1}{2} t \right) \]
(a) Gravitational field: \( \mathbf{F} = -\frac{GM}{r^2} \)

Surface integral: \( \int_S d\mathbf{A} = -4\pi r^2 (\text{Newt. Law}) \)

(b) For \( r = R \):
M\text{enclosed} = \rho \cdot \left( \frac{4}{3} \pi R^3 \right)
\[ \rho = \frac{M}{4\pi R^3} \]
\[ M\text{enclosed} = M \left( \frac{3}{R^3} \right) \]

From Gauss' Law: \( \int_S \mathbf{F} \cdot d\mathbf{A} = -4\pi \rho \int_V \mathbf{F} \cdot d\mathbf{V} \)
\[ 4\pi R^2 \mathbf{F} \cdot \mathbf{F} = -4\pi \rho \int_V \mathbf{F} \cdot d\mathbf{V} \]
\[ \rho \mathbf{F} = -\frac{GM}{r^2} \]

For \( r > R \):
M\text{enclosed} = M

From Gauss' Law: \( \int_S \mathbf{F} \cdot d\mathbf{A} = -4\pi \rho \int_V \mathbf{F} \cdot d\mathbf{V} \)
\[ 4\pi R^2 \mathbf{F} \cdot \mathbf{F} = -4\pi \rho \int_V \mathbf{F} \cdot d\mathbf{V} \]
\[ \rho \mathbf{F} = -\frac{GM}{r^2} \]

For \( r = R \):
\[ \mathbf{F}(r = R) = -\frac{GM}{R^2} \]

Plot:
\[ \mathbf{F} \]
\[ \begin{array}{c}
\text{at } r = R \\
\mathbf{F}(r = R) = -\frac{GM}{R^2}
\end{array} \]
\[ \begin{align*}
\text{for } t > R, \quad & u(t) = g \frac{M}{R^3} \left( \frac{R^3}{2t^2} \right) + C \\
\text{for } t = R, \quad & u(t) = -g \frac{M}{R} \\
& \left( u(t \to \infty) = 0 \right)
\end{align*} \]

Apply continuity condition at \( t = R \):

\[ g \frac{M}{R^3} \left( \frac{R^3}{2} \right) + C = -g \frac{M}{R} \]

\[ C = g \frac{M}{R} \left( \frac{R}{2} \right) \]

\[ u(t) = \begin{cases} 
\frac{g M}{R^3} \left[ \frac{R^3}{2} - \frac{3R^3}{2t^2} \right] & \text{for } t \leq R \\
-g \frac{M}{R} \left( \frac{1}{t} \right) & \text{for } t > R \\
-\frac{g M}{R} & \text{for } t = 0
\end{cases} \]
Part I—Mechanics

1-5 Special Relativity (10)

A stick of lengths \( l \) is fixed at an angle \( \theta \) from its \( x_1 \)-axis in its own rest system \( K \).

What is the lengths and orientation of the stick as measured by an observer moving along \( x_1 \) with relativistic speed \( v \)?

\[ l \cos \theta \sqrt{1 - \frac{v^2}{c^2}} \]

while the perpendicular component will be unchanged:

\[ l \sin \theta \]

So, to the observer in \( K \), the length and orientation of the stick are:

\[ l' = l \left[ \sin^2 \theta + \left( \frac{1 - v^2/c^2}{\cos^2 \theta - v^2/c^2} \right) \right]^{1/2} \]

\[ \theta' = \tan^{-1} \left[ \frac{\sin \theta}{\cos \theta \sqrt{1 - v^2/c^2}} \right] \]

or

\[ l' = l \left[ \sin^2 \theta + \frac{\cos^2 \theta}{\gamma^2} \right]^{1/2} \]

\[ \tan \theta' = \gamma \tan \theta \]
Suppose that the region \( z > 0 \) in three-dimensional space is filled with a linear dielectric material characterized by a dielectric constant \( \varepsilon_1 \), while the region \( z < 0 \) has a dielectric material \( \varepsilon_2 \). Fix a charge \(-q\) at \((x,y,z) = (0,0,a)\) and a charge \(+q\) at \((0,0,-a)\). What is the force one must exert on the negative charge to keep it at rest?

**Solution:**

Consider first the case where a point charge \( q_1 \) is placed at \((0,0,a)\). The method of images requires image charges \( q_1' \) at \((0,0,-a)\) and \( q_1'' \) at \((0,0,a)\). Then the potential (in Gaussian units) at a point \((x,y,z)\) is given by

\[
\phi_1 = \frac{q_1}{\varepsilon_1 \sqrt{\frac{1}{r_1^2}}} + \frac{q_2}{\varepsilon_2 \sqrt{\frac{1}{r_2^2}}}, \quad (z \geq 0),
\]

\[
\phi_2 = \frac{q_2}{\varepsilon_2 \sqrt{\frac{1}{r_2^2}}}, \quad (z < 0),
\]

where \( r_1 = \sqrt{x^2 + y^2 + (z-a)^2} \), \( r_2 = \sqrt{x^2 + y^2 + (z+a)^2} \).

Applying the boundary conditions at \((x,y,0)\):

\[
\phi_1 = \phi_2, \quad \varepsilon_1 \frac{d\phi_1}{dz} = \varepsilon_2 \frac{d\phi_2}{dz}
\]

we obtain:

\[
q_1 = q_1', \quad \frac{(\varepsilon_1 - \varepsilon_2)}{\varepsilon_1 (\varepsilon_1 + \varepsilon_2)} q_1
\]

Similarly, for a point charge \( q_2 \) placed at \((0,0,-a)\) inside the dielectric \( \varepsilon_2 \), its image charges will be \( q_2' \) at \((0,0,a)\) and \( q_2'' \) at \((0,0,-a)\) with magnitudes

\[
q_2' = -\varepsilon_2 \varepsilon_1 \frac{q_2}{\varepsilon_1 (\varepsilon_1 + \varepsilon_2)} q_2.
\]

When both \( q_1 \) and \( q_2 \) exist, the force on \( q_1 \) will be the resultant due to \( q_2, q_1' \), and \( q_1'' \). It follows that

\[
F = \frac{q_1 q_2}{4\varepsilon_0^2 r_1^2} + \frac{q_1 q_2'}{4\varepsilon_0^2 r_2^2} + \frac{q_1 q_2''}{4\varepsilon_0^2 r_2^2} = \frac{\varepsilon_1 - \varepsilon_2}{\varepsilon_1 (\varepsilon_1 + \varepsilon_2)} q_1^2/4\varepsilon_0^2 + q_1 q_2/2(\varepsilon_1 + \varepsilon_2) a^2.
\]

For the current problem, \( q_1 = -q \) and \( q_2 = +q \), one has

\[
F = \frac{q^2}{\varepsilon_1 (\varepsilon_1 + \varepsilon_2) 4\varepsilon_0^2} + \frac{q^2}{2(\varepsilon_1 + \varepsilon_2) a^2} = \frac{q^2}{4\varepsilon_1 a^2}.
\]

Hence, a force of \(-F\) is required to keep \(-q\) at rest.
\begin{align*}
\text{(1)} & \quad m \ddot{v}(t) = m g \sin \theta - I(t) / R \cos \theta \\
\text{(2)} & \quad \ddot{v}(t) / R \cos \theta = L I(t) = 0 \quad \text{(Newton's II.)} \\
& \quad m \ddot{v} = -I(t) / R \cos \theta = -v(t) / R \cos \theta \\
& \quad m \ddot{v} = -\frac{e^2 B^2 \cos \theta}{L} v(t) \\
& \quad \ddot{v} = -\frac{e^2 B^2 \cos \theta}{L m} v(t) \\
\text{(3)} & \quad \begin{cases} 
\dot{v}(t) = A \cos(\omega t) + B \sin(\omega t) \\
\dot{v}(t) = -\omega A \sin(\omega t) + \omega B \cos(\omega t) 
\end{cases} \\
\text{need } \dot{v}(0) \text{ and } \dddot{v}(0) \\
\dot{v}(0) = 0 \\
\text{and from } I(t) = 0 \text{ and (1): } m \dddot{v}(0) = m g \sin \theta = 0 \quad \Rightarrow \dddot{v}(0) = 0 \\
t = 0; \text{ from (3): } \quad \omega = A \\
9 \sin \theta = \omega^2 B \quad \Rightarrow \quad B = \frac{9 \sin \theta}{\omega^2} = \frac{9 m L}{e^2 B^2 \cos \theta} \frac{1}{L} \\
\mu(t) = \frac{9 \sin \theta \sqrt{m L}}{e^2 B \cos \theta} \sin \left( \frac{e B \cos \theta}{\sqrt{m L}} t \right) 
\end{align*}
For a single current loop, located at position \( z = \zeta \), the B-field at point \( \zeta \):

\[
F_z = \frac{\mu_0 I}{2} \frac{R^2}{(R^2 + \zeta^2)^{3/2}}
\]

\( F_x = F_y = 0 \) (By symmetry)

The current has \( N \) turns per unit length:

\[
dI = (d\zeta) N / I
\]

\[
dB_z = \frac{\mu_0 N I}{2} \frac{d\zeta}{(R^2 + \zeta^2)^{3/2}}
\]

\[
B_z = \frac{\mu_0 N I}{2} \int_{\zeta_1}^{\zeta_2} \frac{R^2 d\zeta}{(R^2 + \zeta^2)^{3/2}}
\]

\[
B_z = \frac{\mu_0 N I}{2} \frac{R^2 \zeta_2}{\sqrt{R^4 - \zeta_2^2}}
\]
Solution:

\[ I - 9 \]

\[ \vec{F} = \vec{x} \times \vec{E} \]

\( d \)

\[ \vec{E}_{dc} = \frac{1}{\mu \mu_0} \frac{\sigma_{dc} \vec{E}_E}{\mu_a} \]

\[ \vec{n}_{dc} = \frac{\vec{n} \times \vec{E}_{dc}}{\mu_a} \]

\[ \vec{v} = \frac{(\vec{F} \times \vec{v})}{4\pi \epsilon_0 \mu_0} \]

\[ \vec{U} = \frac{\vec{F} \times \vec{V}}{4\pi \epsilon_0 \mu_0} \]

\[ F = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \]

\[ \vec{E} = \vec{n} \times \vec{n} = \vec{0} \]

\[ \vec{E}_E = \vec{n} \times \vec{n} = \vec{0} \]

\[ \vec{v}_{dc} = \vec{0} \]

\[ \vec{u}_{dc} = \vec{0} \]

\[ \mu \mu_0 = \frac{1}{\sigma_{dc}} \]

\[ \vec{E}_{dc} = \vec{0} \]

\[ \vec{n} \times \vec{n} = \vec{0} \]

\[ \vec{E}_E = \vec{0} \]

\[ \vec{v} = \vec{0} \]

\[ \vec{u} = \vec{0} \]

\[ \vec{n} \times \vec{n} = \vec{0} \]

\[ \vec{E}_{dc} = \vec{0} \]

\[ \vec{v}_{dc} = \vec{0} \]

\[ \vec{u}_{dc} = \vec{0} \]

\( b \)

\[ L = \frac{1}{2} \mu \epsilon_0 \left( \frac{1}{2} \mu \epsilon_0 \right) \]

\[ S = \frac{\mu^2 \epsilon_0^2 \mu_0^2}{12\pi^2} \]

\[ \frac{\mu^2 \epsilon_0^2 \mu_0^2}{12\pi^2} \]
let the field be \( \mathbf{E} \) in frame S:

let the field be \( \mathbf{E}' \) in frame S':

let \( \gamma = \frac{1}{\sqrt{1-v^2}} \)

we have:

\[
\begin{align*}
E_x &= E'_x \\
E_y &= \gamma (E_y + \beta c E'_z) \\
E_z &= \gamma (E_z - \beta c E'_y)
\end{align*}
\]

\[
\begin{align*}
B_x &= B'_x \\
B_y &= \gamma (B_y - \frac{\beta}{c} E'_z) \\
B_z &= \gamma (B_z + \frac{\beta}{c} E'_y)
\end{align*}
\]

Now in the frame S':

\( B'_x = B'_y = B'_z = 0 \)

\( E'_x = E'_y = 0 \)

\( E'_z = \frac{\gamma}{E_0} \)

\[
\begin{align*}
\gamma E_0 &= E'_0 \\
\gamma E_y &= \frac{\gamma}{c} (E'_y) \\
\gamma E_z &= \frac{\gamma}{c} (E'_z)
\end{align*}
\]
In the case of face: 
\[ F_x = E_y = E_z = 0 \]
\[ E_x = 0 \]
\[ E_y = E_z = \frac{J_0}{2 \pi r^2} \]

\[ E_x = 0, \quad E_y = J \left( \frac{J_0}{2 \pi r^2} \right) \]
\[ E_z = J \left( \frac{J_0}{2 \pi r^2} \right) \]

\[ B_x = 0, \quad B_y = -\frac{J_0}{2} \left( \frac{J_0}{2 \pi r^2} \right), \quad B_z = \frac{J_0}{2} \left( \frac{J_0}{2 \pi r^2} \right) \]
Physics PhD Qualifying Examination
Part II – Friday, January 17, 2014

Name: ____________________________
(please print)

Identification Number: ____________

STUDENT: Insert a check mark in the left boxes to designate the problem numbers that you are handing in for grading.

PROCTOR: Check off the right hand boxes corresponding to the problems received from each student. Initial in the right hand box.

<p>| | | | | | | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>9</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>10</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Student’s initials
# problems handed in:
Proctor’s initials

INSTRUCTIONS FOR SUBMITTING ANSWER SHEETS

1. DO NOT PUT YOUR NAME ON ANY ANSWER SHEET. EXAMS WILL BE COLLATED AND GRADED BY THE ID NUMBER ABOVE.

2. Use at least one separate preprinted answer sheet for each problem. Write on only one side of each answer sheet.

3. Write your Identification number listed above, in the appropriate box on the preprinted sheets.

4. Write the problem number in the appropriate box of each preprinted answer sheet. If you use more than one page for an answer, then number the answer sheets with both problem number and page (e.g. Problem 9 - Page 1 of 3).

5. Staple together all the pages pertaining to a given problem. Use a paper clip to group together all problems that you are handing in.

6. A passing distribution for the individual components will normally include at least four passed problems (from problems 1-6) for Quantum Physics and two problems (from problems 7-10) for Thermodynamics and Statistical Mechanics.

7. **YOU MUST SHOW ALL YOUR WORK.**
A quantum mechanical system is known to possess only two energy eigenstates denoted \(|1\rangle\) and \(|2\rangle\). The system also includes three other observables (besides the energy), known as \(P\), \(Q\), and \(R\). The states denoted \(|1\rangle\) and \(|2\rangle\) are normalized but they are not necessarily eigenstates of \(P\), \(Q\), or \(R\). A series of experiments are performed yielding,

Experiment 1: \(\langle 1 | P | 1 \rangle = 1/2\) and \(\langle 1 | P^2 | 1 \rangle = 1/4\)

Experiment 2: \(\langle 1 | Q | 1 \rangle = 1/2\) and \(\langle 1 | Q^2 | 1 \rangle = 1/6\)

Experiment 3: \(\langle 1 | R | 1 \rangle = 1\), \(\langle 1 | R^2 | 1 \rangle = 5/4\), and \(\langle 1 | R^3 | 1 \rangle = 7/4\)

(a) One set of experimental data is erroneous and yields unphysical results. Which dataset is it?

(b) Determine as many eigenvalues of \(P\), \(Q\), and \(R\) as possible from the information given.

A particle moves in a one-dimensional box with small potential hump in half of the well, as shown in the figure.

\[
V(x) = \begin{cases} 
\infty & (x < 0, x > L) \\
0 & (x > 0, x < \frac{L}{2}) \\
b & (x > \frac{L}{2}, x < L)
\end{cases}
\]

Treating the potential hump as a perturbation to the "regular" infinite potential well problem, with \(V(x) = \infty \) \((x < 0, x > L)\) and \(V(x) = 0 \) \((0 < x < L)\), determine the first order energy of the ground state.
A beam of identically prepared spin \( \frac{1}{2} \) atoms with \( S_z = +\hbar/2 \) orientation and with unit intensity goes through a series of Stern-Gerlach-type (SG) measurements (selective filtering) as follows:

- The first measurement accepts \( S_n = +\hbar/2 \) atoms (and rejects \( S_n = -\hbar/2 \) ones), where \( S_n \) is the eigenvalue of the spin operator along the direction \( \hat{n} = (\cos \phi \sin \theta, \sin \phi \sin \theta, \cos \theta) \), \( \hat{S} \cdot \hat{n} \). (\( \theta \) and \( \phi \) are the polar and azimuthal and angles, respectively).
- The second measurement accepts \( S_z = -\hbar/2 \) atoms (and rejects \( S_z = +\hbar/2 \) ones).

What is the intensity of the final \( S_z = -\hbar/2 \) beam? (You must express your answer in terms of \( \theta \) and \( \phi \).)

| II-4 | [10] |

(a) Consider a potential of the form \( V(r) = V_o \frac{e^{-r}}{r} \).

(i) Calculate the differential scattering cross section using Born approximation.
(ii) Calculate the total scattering cross section.

(b) Using the results of (a), calculate (i) the differential and (ii) the total scattering cross section of Coulomb potential \( V_C(r) = \frac{Z_i Z_f e^2}{r} \)
(a) Write down uncertainty relationship of space and momentum;
(b) Use the uncertainty relationship to estimate the ground-state energy of a particle, mass \( m \), in a one-dimensional infinite potential well of width \( L \);
(c) Use the uncertainty relationship to estimate the ground-state energy of a one-dimensional simple harmonic oscillator of mass \( m \) and oscillating angular-frequency \( \omega \).

\[ \Psi(x, t) \] is a solution of the time-dependent Schrödinger equation

\[
i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m} \Delta \Psi + V(x) \Psi.\]

Show that \( \rho(x, t) = |\psi(x, t)|^2 \) and \( j(x, t) = \frac{i\hbar}{2m} \left( \nabla \psi \psi^* - \psi^* \nabla \psi \right) \) satisfy the continuity equation

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot j = 0.
\]
A gas is initially confined to one-half of a thermally isolated container. The other half is empty. The gas is suddenly permitted to expand to fill the entire chamber ((Gay-Lussac — Joule) "free expansion"). Assuming the initial temperature of the gas in the half-container is \( T_1 \), find the temperature \( T_f \) after the expansion for the following two cases:

(a) Equation of state: \( pV = nRT \)

(b) Equation of state: \( b \left(p + \frac{a}{V^2}\right) = nRT \)

\( a \) and \( b \) are positive constants, \( p, V \) and \( T \) denote pressure, volume and temperature, respectively. \( R \) is the universal gas constant and \( n \) is the mol number.

---

The Helmholtz free energy of a dilute plasma gas consisting of \( N \) electrons confined to a volume \( V \) at temperature \( T \) is given by

\[
F(T, V, N) = F_{\text{ideal}}(T, V, N) - \frac{2}{3} N e^2 \left( \frac{4 \pi N e^2}{V k_B T} \right)^{1/3}
\]

where \( e \) is the electron charge, \( k_B \) is Boltzmann's constant, and \( F_{\text{ideal}}(T, V, N) \) is the Helmholtz free energy of a mono-atomic ideal gas.

(a) Obtain the equation of state of the above plasma gas.

(b) Obtain the internal energy of the system \( E(T, V, N) \).

(c) Obtain the constant-volume heat capacity \( C_V \).

Note: your answers should be fully explicit in terms of the variables \( T, V, \) and \( N \). To that end, you are expected to remember and use the equation of state, internal energy, heat capacity of the ideal gas in obtaining your final results.
In the Einstein model for a three-dimensional solid of \( N \) atoms, the system is treated as an ensemble of \( 3N \) distinguishable and independent quantum harmonic oscillators with identical frequency \( \omega_o \), (three oscillators for each atoms).

(a) Find the specific heat of this simple model for solids (the Einstein crystal). You must express your answer in terms of \( N, k_B \) (Boltzmann’s constant), and the ratio \( \theta_E/T \), where \( \theta_E = h\omega_o / k_B \) is the characteristic Einstein temperature.

(b) Obtain the high-temperature behavior of the specific heat. How does it compare to the classical behavior?

(c) Obtain the low-temperature behavior of the specific heat. What can you say about its limiting value as \( T \rightarrow 0 \) ?

Consider a three-dimensional extreme-relativistic \( (c = \omega) \) free electron gas confined to a volume \( V = L^3 \). The number of electrons is \( N \).

Obtain \( P \), the pressure of the system at \( T = 0 \). You must express your answer in terms of the electron (number) density \( N/V \).
A quantum mechanical system is known to possess only two energy eigenstates denoted \( |1\) and \( |2\). The system also includes three other observables (besides the energy), known as \( P, Q, \) and \( R. \) The states denoted \( |1\) and \( |2\) are normalized but they are not necessarily eigenstates of \( P, Q, \) or \( R. \) A series of experiments are performed yielding:

Experiment 1: \( \langle 1|P|1\rangle = 1/2 \) and \( \langle 1|P^2|1\rangle = 1/4 \)

Experiment 2: \( \langle 1|Q|1\rangle = 1/2 \) and \( \langle 1|Q^2|1\rangle = 1/6 \)

Experiment 3: \( \langle 1|R|1\rangle = 1, \) \( \langle 1|R^2|1\rangle = 5/4, \) and \( \langle 1|R^3|1\rangle = 7/4. \)

(a) One set of experimental data is erroneous and yields unphysical results. Which dataset is it?

(b) Determine as many eigenvalues of \( P, Q, \) and \( R \) as possible from the information given.

Solution:

(a) The results from experiment 2 are unphysical. Writing \( Q \) as a Hermitian operator, \( Q = \begin{pmatrix} a & b \\ b^* & c \end{pmatrix} \)
the results of the experiment can be written as:

\[
\langle 1|Q|1\rangle = (1 \ 0) \begin{pmatrix} a & b \\ b^* & c \end{pmatrix} \left( \begin{array}{c} 1 \\ 0 \end{array} \right) = a = \frac{1}{2}
\]

and,

\[
\langle 1|Q^2|1\rangle = (1 \ 0) \begin{pmatrix} a & b \\ b^* & c \end{pmatrix} \begin{pmatrix} a & b \\ b^* & c \end{pmatrix} \left( \begin{array}{c} 1 \\ 0 \end{array} \right) = (1 \ 0) \begin{pmatrix} a^2 + b b^* \\ b^* a + c c \end{pmatrix} = \frac{1}{6}
\]

The result \( \langle 1|Q|1\rangle = \frac{1}{2} \) requires \( \alpha = \frac{1}{2}. \) However, from \( \langle 1|Q^2|1\rangle = \frac{1}{6} \) we see that no \( \beta \) can yield the "observed" results, as this requires \( \beta \beta^* = -\frac{1}{12}. \)

(b) Operator \( P: \)

\[
\begin{align*}
(1|P|1) &= 1/2 \Rightarrow \alpha = 1/2 \\
(1|P^2|1) &= 1/4 \Rightarrow \alpha^2 + \beta \beta^* = \frac{1}{4}, \ \beta = 0
\end{align*}
\]

Therefore, \( P = \begin{pmatrix} 1/2 & 0 \\ 0 & 0 \end{pmatrix}, \) where \( \gamma \) cannot be determined. It is already diagonal, and hence:

\[
\lambda_1 = \frac{1}{2}, \quad \lambda_2 = \gamma = \text{unknown}
\]
Operator $R$:

(i) $\langle 1 | R | 1 \rangle = 1 \rightarrow \alpha = 1$

(ii) $\langle 1 | R^2 | 1 \rangle = 5/4 \rightarrow \alpha^2 + \beta \beta^* = \frac{5}{4}, \beta \beta^* = \frac{1}{4}, \beta = \frac{1}{2} e^{i\delta_0}$

(iii) $\langle 1 | R^3 | 1 \rangle = (1 - 0) \left( \begin{array}{cc} \alpha & \beta \\ \beta^* & \gamma \end{array} \right) \left( \begin{array}{cc} \alpha^2 + \beta \beta^* \\ \beta^* \alpha + \gamma \beta^* \end{array} \right) = (1 - 0) \left( \begin{array}{cc} \alpha (\alpha^2 + \beta \beta^*) + \beta (\beta^* \alpha + \gamma \beta^*) \\ \beta^* (\alpha^2 + \beta \beta^*) + \gamma (\beta^* \alpha + \gamma \beta^*) \end{array} \right) = \\
\alpha (\alpha^2 + \gamma \beta^*) + \beta (\beta^* \alpha + \gamma \beta^*) = \left( 1 + \frac{1}{4} \right) + \frac{1}{4} + \frac{1}{4} = \frac{7}{4}, \gamma = 1.$

Therefore, $R = \left( \begin{array}{cc} 1 & \frac{1}{2} e^{i\delta_0} \\ \frac{1}{2} e^{-i\delta_0} & 1 \end{array} \right)$ and $\left| \begin{array}{cc} 1 - \lambda & \frac{1}{2} e^{i\delta_0} \\ \frac{1}{2} e^{-i\delta_0} & 1 - \lambda \end{array} \right| = (1 - \lambda)^2 - \frac{1}{4} = 0.$

With eigenvalues $\lambda_1 = \frac{1}{2}, \lambda_2 = \frac{3}{2}$.
A particle moves in a one-dimensional box with small potential hump in half of the well, as shown in the figure.

\[
V(x) = \begin{cases} 
\infty & (x < 0, x > L) \\
0 & (x > 0, x < \frac{L}{2}) \\
\beta & (x > \frac{L}{2}, x < L)
\end{cases}
\]

Treating the potential hump as a perturbation to the "regular" infinite potential well problem (\(V(x) = \infty (x < 0, x > L)\); \(V(x) = 0 (0 < x < L)\)), determine the first order energy of the ground state.

**Solution:**

For the ground state of the "regular" infinite potential well we have:

\[
\Psi_0 = A\sin\left(\frac{nx}{L}\right), \text{ with energy } E_0 = \frac{n^2\hbar^2}{2mL^2}.
\]

The perturbation \(V'(x) = \begin{cases} 
\frac{\hbar^2}{L} & (0 < x < \frac{L}{2}) \\
\beta & (x > \frac{L}{2})
\end{cases}\)

In the context of first order perturbation theory,

\[
E = E_0 + \langle\psi_0|H'|\psi_0\rangle
\]

\[
= E_0 + \int_0^{\frac{L}{2}} V'(x)A^2\sin^2\frac{nx}{L} dx - E_0 + \int_0^{\frac{L}{2}} A^2 \sin^2\frac{nx}{L} dx + \int_{\frac{L}{2}}^L \beta A^2 \sin^2\frac{nx}{L} dx
\]

\[
= E_0 + b \int_0^{\frac{L}{2}} A^2 \sin^2\frac{nx}{L} dx = E_0 + \frac{\hbar^2}{2mL^2}
\]

Therefore, \(E = \frac{n^2\hbar^2}{2mL^2} + \frac{\hbar^2}{2mL^2}\).
\[ \frac{I - \Phi}{\Pi \Phi} \left( \begin{array}{c}
|+\rangle = |1_{0}\rangle + |1_{\pi}\rangle \\
|-\rangle = |1_{0}\rangle - |1_{\pi}\rangle 
\end{array} \right) \]

\[ |1_{m_1}\rangle = |\pm \pi m_1, \pm \rangle = \cos \left( \frac{\pi m_1}{2} \right) |+\rangle + \sin \left( \frac{\pi m_1}{2} \right) e^{i\Phi} |-\rangle \]

\[ |1_{m_1}\rangle = |\pm \pi m_1, \mp \rangle = \sin \left( \frac{\pi m_1}{2} \right) |+\rangle - \cos \left( \frac{\pi m_1}{2} \right) e^{i\Phi} |-\rangle \]

\[ \frac{\Phi_1}{\sqrt{2}} = |1_{m_1}, +\rangle |^2 = |1_{+}, 1_{m_1}, +\rangle |^2 = \cos^2 \left( \frac{\pi m_1}{2} \right) \]

\[ \frac{\Phi_2}{\sqrt{2}} = |1_{m_1}, -\rangle |^2 = \sin^2 \left( \frac{\pi m_1}{2} \right) \]

\[ I_{\text{max}} = \frac{\Phi_1 \cdot \Phi_2}{\Phi_1 + \Phi_2} = \cos^2 \left( \frac{\pi m_1}{2} \right) - \sin^2 \left( \frac{\pi m_1}{2} \right) = \frac{1}{4} \left[ 2 \cos \left( \frac{\pi m_1}{2} \right) + \sin \left( \pi m_1 \right) \right] \]
(II-4)

(a) From Born Approximation:

\[ f'(k, k') = \frac{1}{4\pi} \frac{2k}{\hbar^2} \int d^3x \ e^{i(k-k') \cdot x} V(x) \]

(c) \[ |\mathbf{k} - \mathbf{k}'| = 2k \sin \theta/2 \]

\[ |\mathbf{k}| = 2k \sin \theta/2 \]

\[ \sin f'(0) = -\frac{1}{4\pi} \frac{2kV_0}{\hbar^2} \int_0^\infty dt \sin(kt) \frac{e^{-\alpha t}}{\sqrt{t}} \]

\[ = -\frac{2kV_0}{\hbar^2} \frac{1}{\alpha^2 + |\mathbf{k}|^2} \]

\[ = -\frac{2kV_0}{\hbar^2} \left[ \frac{1}{\alpha^2 + 4k^2 \sin^2 \theta/2} \right] \]

\[ \delta(\theta) = \frac{4\mu^2 V_0^2}{\hbar^4} \frac{1}{\left[ \alpha^2 + 4k^2 \sin^2 \theta/2 \right]^2} \]

\[ \delta_{\text{total}} = \int d\theta \delta(\theta) = \frac{4\mu^2 V_0^2}{\hbar^4} \frac{4\pi}{\alpha^2 (\alpha^2 + 4k^2)} \]
(b) For Coulomb potential, $V = \frac{\pm_1 \pm_2 e^2}{r}$

set $V_0 = \pm_1 \pm_2 e^2$ & $\chi = 0$

(i) $\Delta(\theta) = \frac{4/12}{\pi^4} - \frac{\pm_1^2 \pm_2^2 e^4}{16 \pi^4 \sin^2 \phi}$

(ii) $\sigma_{\text{total}} \rightarrow \infty$
(a) The uncertainty relationship: 
\[ \Delta x \Delta p = \frac{\hbar}{2} = \frac{\pi}{2} \]

(b) For an infinite well of width \( L \)

Gives: \( x = L \)

Use uncertainty relationship: 
\[ \Delta x \Delta p \approx \frac{\hbar}{2} \left( \frac{1}{\Delta x} \right) = \frac{\hbar}{2\Delta x} \]

Total energy: 
\[ E_0 = \frac{\hbar^2}{2m} + V = \frac{\hbar^2}{2m} \left[ \frac{L^2}{2L^2} \right] + V \]

\[ E_0 = \frac{\hbar^2}{2mL^2} \]

(c) Total energy of a harmonic oscillator:
\[ E = \frac{p^2}{2m} + \frac{1}{2} kx^2 \]

From uncertainty relationship: 
\[ \Delta x \Delta p = \frac{\hbar}{2} \]

\[ E = \frac{\hbar^2}{8mL^2} + \frac{1}{2} mL^2x^2 \]

To obtain ground state energy, we minimize the above expression.
\[ \frac{\pi^2}{2m} \lambda^2 + \frac{3m}{2} \lambda^2 \omega^2 \lambda = 0 \]

\[ \lambda = \sqrt{\frac{7m}{3m \omega^2}} \]

Ground state energy:

\[ E = \frac{\pi^2}{8m} \lambda^2 + \frac{3m}{2} \lambda \omega^2 \left[ \frac{\pi^2}{8m} \frac{1}{2} \right] \]

\[ E = \frac{\pi^2 \lambda \omega}{8m} + \frac{3m}{2} \lambda \omega = \frac{3m \lambda \omega}{8m} \]
In the above equation $V(x)$ is an Hermitian Operator, $Y(x,t)$ is an eigenfunction of $H$ and the eigenvalues are real.
\[ \text[I-7] \text{Free expansion of a gas} \]

\[ dU = dQ + dW = 0 \quad dQ = 0 \quad \text{adiabatic} \]
\[ dU = T \, ds + pdV \quad dW = 0 \quad \text{ideal gas expansion} \]
\[ U = U(T,V) = \text{const.} \]

\[ c_v = T \left( \frac{\partial s}{\partial T} \right)_V \]
\[ \frac{\partial s}{\partial V} = \left( \frac{\partial p}{\partial T} \right)_V \]
\[ \frac{c_v}{c_v} \, dT + \left( T \left( \frac{\partial p}{\partial T} \right)_V - P \right) dV = 0 \]
\[ dT = \frac{1}{c_v} \left( P - T \left( \frac{\partial p}{\partial T} \right)_V \right) dV \]

(a) \[ pV = nRT \]
\[ p = \frac{nRT}{V} \]
\[ \left( \frac{\partial p}{\partial T} \right)_V = \frac{nR}{V} \quad \Rightarrow \quad dT = \frac{1}{c_v} \left( \frac{P - T \frac{nRT}{V}}{V} \right) dV = 0 \]
\[ dT = 0 \quad \Rightarrow \quad T_F = T_i \]

(b) \[ b \left( p + \frac{a}{V^2} \right) = nRT \]
\[ p = \frac{nRT}{b} - \frac{a}{V^2 b} \]
\[ \left( \frac{\partial p}{\partial T} \right)_V = \frac{nR}{b} \quad \Rightarrow \quad dT = \frac{1}{c_v} \left[ \frac{nRT}{b} - \frac{a}{V^2} - \frac{T \frac{nRT}{b}}{V^2} \right] dV \]
\[ dT = - \frac{1}{c_v} \frac{a}{V^2} dV \]
\[ \int_{T_i}^{T_f} dT = - \frac{1}{c_v} \int_{V_i}^{V_f} \frac{dV}{V^2} \quad V_i = V_0 \quad V_F = 2V_0 \]
\[ T_F - T_i = \frac{a}{c_v} \left( \frac{1}{V_F} - \frac{1}{V_i} \right) = \frac{-a}{c_v V_0} \]
\( T(T, V, N) = T_{\text{ideal}}(T, V, N) - \frac{2}{5} N_{e} \left( \frac{47N_{e}}{V_{ab}} \right) \frac{1}{\sqrt{V}} \)

\[ \alpha \] \( p = \frac{\partial T}{\partial V_{T, N}} = \frac{\partial T_{\text{ideal}}}{\partial V_{T, N}} - \frac{2}{5} N_{e} \left( \frac{1}{2N_{e}^{2}} \right) \left( \frac{47N_{e}}{V_{ab}} \right) \frac{1}{\sqrt{V}} \)

\( P(i, V) = \frac{N_{e} T_{ab}}{V} - \frac{1}{3} \frac{N_{e}^{2}}{V} \left( \frac{47N_{e}}{V_{ab}T} \right) \)

\[ b \] \( F = F - TS \)

\( E = F + TS \) and \( S = -\frac{\partial F}{\partial T} \)

\[ S = \frac{\partial T}{\partial V_{T, N}} = \frac{\partial T_{\text{ideal}}}{\partial V_{T, N}} - \frac{2}{5} N_{e} \left( \frac{1}{2N_{e}^{2}} \right) \left( \frac{47N_{e}}{V_{ab}T} \right) \frac{1}{\sqrt{V}} \]

\[ E = F + TS = F_{\text{ideal}} - \frac{3}{5} N_{e} \left( \frac{47N_{e}}{V_{ab}T} \right) \frac{1}{\sqrt{V}} + TS_{\text{ideal}} - \frac{1}{5} N_{e} \frac{1}{\sqrt{V}} \]

\[ \frac{E}{V} = \frac{F_{\text{ideal}}}{V} + TS_{\text{ideal}} - \frac{1}{5} N_{e} \frac{1}{\sqrt{V}} \frac{1}{\sqrt{V}} \]

\[ C_{e} = \frac{\partial E}{\partial V_{T, N}} = \frac{3}{2} N_{e} - \frac{1}{5} N_{e} \left( \frac{47N_{e}}{V_{ab}T} \right) \frac{1}{\sqrt{V}} \]

After possible rearranging: \( C_{e} \):

\[ C_{e} = \frac{\delta Q}{\delta T} = T \frac{\partial E}{\partial T} = T \left( \frac{\partial F_{\text{ideal}}}{\partial T} \right) \]

\[ = -T \left\{ \frac{\partial F_{\text{ideal}}}{\partial V_{T, N}} - \frac{2}{5} \frac{1}{2} \frac{3}{2} N_{e} \left( \frac{47N_{e}}{V_{ab}T} \right) \frac{1}{\sqrt{V}} \right\} \]

\[ = C_{e, \text{ideal}} + \frac{N_{e}^{2}}{2T} \left( \frac{47N_{e}}{V_{ab}T} \right) \frac{1}{\sqrt{V}} \]

\[ = \frac{3}{2} N_{e} + \frac{N_{e}^{2}}{2T} \left( \frac{47N_{e}}{V_{ab}T} \right) \frac{1}{\sqrt{V}} \]
ensemble of $3N$ distinguishable independent quantum oscillators, $E_i = \hbar \omega_0 (n_i + \frac{1}{2})$, $n_i = 0, 1, 2, \ldots$

\[ Z = Z_{3N} \]

\[ Z = \sum_{n=0}^{\infty} e^{-\beta \hbar \omega_0 (n + \frac{1}{2})} = e^{-\frac{\hbar \omega_0}{2}} \sum_{n=0}^{\infty} e^{-\beta \hbar \omega_0 n} = e^{-\frac{\hbar \omega_0}{2}} \frac{1}{1 - e^{-\beta \hbar \omega_0}} \]

\[ \langle E \rangle = -\frac{2}{\beta} \ln Z_{3N} = -3N \frac{2}{\beta} \text{ln} Z_1 \]

\[ \frac{2}{\beta} \text{ln} Z_1 = \frac{2}{\beta} \left( -\frac{\hbar \omega_0}{2} - \text{ln} \left( 1 - e^{-\beta \hbar \omega_0} \right) \right) = -\frac{\hbar \omega_0}{2} - \frac{e^{-\beta \hbar \omega_0}}{1 - e^{-\beta \hbar \omega_0}} \]

\[ \langle E \rangle = \frac{3N \hbar \omega_0}{2} + \frac{3N \hbar \omega_0}{\beta \hbar \omega_0} \quad \text{internal energy of} \quad E = 1 \]

\[ C_N = \frac{\langle 2E \rangle}{T} - \frac{\langle E \rangle}{T^2} = \frac{\langle 2E \rangle}{T} - \frac{1}{T^2} \frac{\langle 3E \rangle}{N} = \frac{3N \hbar \omega_0}{k_B T^2} \frac{\hbar \omega_0 e^{-\beta \hbar \omega_0}}{\left( e^{\beta \hbar \omega_0} - 1 \right)^2} \]

\[ = 3N k_B \left( \frac{\hbar \omega_0}{k_B T} \right)^2 \frac{e^{\frac{\hbar \omega_0}{k_B T}}}{\left( e^{\frac{\hbar \omega_0}{k_B T}} - 1 \right)^2} \]

\[ = 5 N k_B \left( \frac{\Theta}{T} \right)^2 \frac{e^{\Theta / T}}{\left( e^{\Theta / T} - 1 \right)^2} \]

where \( \Theta = \frac{\hbar \omega_0}{k_B} \)
b) high temperature limit \( \frac{\Theta_e}{T} \ll 1 \)

\[
\frac{\Theta_e}{T} \approx 1 + \frac{\Theta_e}{T} + \ldots
\]

\[
C_N \approx 3Nk_b \left( \frac{\Theta_e}{T} \right)^2 \frac{1 + \frac{\Theta_e}{T} + \ldots}{(1 + \frac{\Theta_e}{T} + \ldots - 1)^2}
\]

\[
= \frac{3Nk_b}{(\Theta_e/T)^2} 1 + \frac{\Theta_e}{T} + \ldots = 3Nk_b + O \left( \frac{\Theta_e}{T} \right)
\]

i.e. for high temperatures \( C_N \) approaches a constant

Dulong-Petit Law

classical behavior

2N classical oscillators

equation for \( \Theta_e \)

\[ 2Nk_bT = \langle E \rangle \text{ per oscillators} \]

\[ 2Nk_bT = \langle E \rangle \text{ per oscillator} \]

\[ C_N \approx \frac{\Theta_e}{T} - 3Nk_bT \]

c) low temperature behavior \( \frac{\Theta_e}{T} \gg 1 \)

\[
C_N \approx 3Nk_b \left( \frac{\Theta_e}{T} \right)^2 e^{-\Theta_e/T}
\]

non-analytic at \( T = 0 \)

exponential vanishes much faster than power law divergence at \( T \to 0 \)

This

\[ C \quad T \to 0 \quad \to 0 \text{ specific heat vanishes} \]
\[ \epsilon = c \rho = c \frac{k}{k} = c \frac{\mu}{\lambda} \left( \frac{u^2_1 + u^2_2}{2} \right) \]

\[ \mathbf{k} = \mathbf{k}_\perp = \mathbf{k}_x + \mathbf{k}_y \]

\[ \lambda = \frac{\mu}{\lambda} \]

\[ \epsilon = \frac{c h}{2 L} \left( \frac{u^2_1 + u^2_2}{2} \right)^{\frac{1}{2}} = \frac{c h}{2 L} n \]

\[ \nabla (\epsilon) = \frac{1}{8} \frac{\partial^2 \mathbf{u}}{\partial \mathbf{r}^2} \]

\[ = \frac{1}{8} \frac{\partial^2 \mathbf{u}}{\partial \mathbf{r}^2} \]

\[ y(x) = \frac{1}{2} \left( 2 \epsilon + \frac{\partial^2 \mathbf{u}}{\partial \mathbf{r}^2} \right) \]

\[ T = 0 \]

\[ N = \int_0^{\frac{\pi}{2}} \frac{\mathbf{E}^2}{\mathbf{E}^2} \, d\theta = \frac{8 \pi}{c^2 \mu h^3} \epsilon \]

\[ U_0 = \int_0^{\frac{\pi}{2}} \frac{\mathbf{E}^2}{\mathbf{E}^2} \, d\theta = \frac{8 \pi}{c^2 \mu h^3} \epsilon \]