solutions

Physics PhD Qualifying Examination
Part I – Wednesday, January 15, 2014

Name:______________________________________
(please print)
Identification Number:________

STUDENT: Designate the problem numbers that you are handing in for grading in the appropriate left hand boxes below. Initial the right hand box.
PROCTOR: Check off the right hand boxes corresponding to the problems received from each student. Initial in the right hand box.

	1	
	2	
	3	
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	5	
	6	
	7	
	8	
	9	
	10	

Student's initials
problems handed in:
Proctor's initials

INSTRUCTIONS FOR SUBMITTING ANSWER SHEETS

1. DO NOT PUT YOUR NAME ON ANY ANSWER SHEET. EXAMS WILL BE COLLATED AND GRADED BY THE ID NUMBER ABOVE.
2. Use at least one separate preprinted answer sheet for each problem. Write on only one side of each answer sheet.
3. Write your **identification number** listed above, in the appropriate box on each preprinted answer sheet.
4. Write the **problem number** in the appropriate box of each preprinted answer sheet. If you use more than one page for an answer, then number the answer sheets with both problem number and page (e.g. Problem 9 – Page 1 of 3).
5. Staple together all the pages pertaining to a given problem. Use a paper clip to group together all problems that you are handing in.
6. A passing distribution for the individual components will normally include at least three passed problems (from problems 1-5) for Mechanics and three problems (from problems 6-10) for Electricity and Magnetism.
7. **YOU MUST SHOW ALL YOUR WORK.**

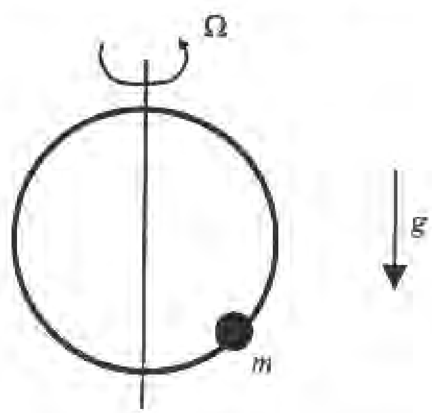
[I-1]   [10]

A particle of mass m is bound by a linear potential $U(r) = kr$.

(a) For what energy and angular momentum will the orbit be a circle of radius r about the origin?
(b) What is the frequency of this circular motion?
(c) If the particle is slightly disturbed from this circular motion, what will be the frequency of small oscillations?

[I-2]   [1,1,2,6]

A particle of mass m is constrained to move (without friction) on a circular wire with radius a, which is rotating with a *fixed* angular velocity Ω about its vertical diameter in the presence of uniform gravity g.



(a) Find a convenient generalized coordinate and write down the Lagrangian of this system.
(b) Obtain the equation of motion.
(c) Find *all* possible equilibrium positions of the particle along the circular wire.
(d) Discuss the *stability* of the above equilibrium points depending on the value of the angular velocity Ω and obtain the *frequency of small oscillations* about the respective stable equilibrium position.

[I-3] **[10]**

Consider the coupled mass-spring system sketched below. The masses can only move horizontally. The springs are relaxed at equilibrium. Obtain the *complete* solution of the problem for the following set of initial conditions:

$$x_1(0) = D,\ x_2(0) = 0;\ \dot{x}_1(0) = 0,\ \dot{x}_2(0) = 0.$$

Note that simply finding the normal frequencies and normal modes is *not* sufficient for obtaining a passing score for this problem. You have to obtain $x_1(t)$ and $x_2(t)$ for the given initial conditions. (The displacements $x_1(t)$ and $x_2(t)$ are measured from the respective equilibrium positions of the masses.)

[I-4] **[10]**

Consider earth with a total mass M and radius R. Also, assume that earth's mass is distributed uniformly.

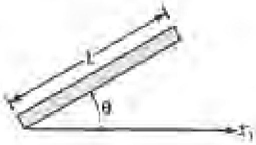
(a) Argue that one may use the equivalent of "Gauss law" to determine gravity inside and outside of the earth's surface. State the "Gauss law".

(b) Derive functional form of gravity g as a function of radial distance r inside and outside the earth's surface. Sketch g vs r. (should mark relevant quantities and functional dependence.)

(c) Derive gravitational potential $U(r)$ as a function of r inside and outside the earth's surface. Sketch U vs r. (should mark relevant quantities and functional dependence.)

[1-5] [10]

A stick of lengths ℓ is fixed at an angle θ from its x_l-axis in its own rest system K.
What is the lengths and orientation of the stick as measured by an observer moving along x_l with relativistic speed v?

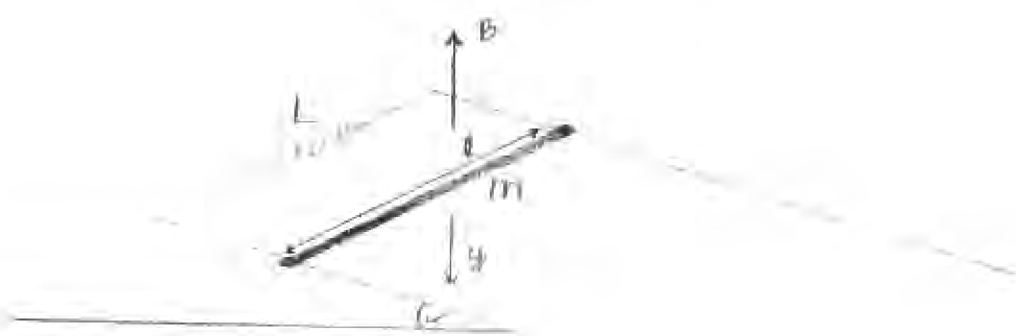


[I-6] [10]

Suppose that the region $z > 0$ in three-dimensional space is filled with a linear dielectric material characterized by a dielectric constant ϵ_1, while the region $z < 0$ has a dielectric material ϵ_2. Fix a charge $-q$ *at* $(x, y, z) = (0,0,a)$ and a charge $+q$ *at* $(0,0,-a)$. What is the force one must exert on the negative charge to keep it at rest?

[I-7] [10]

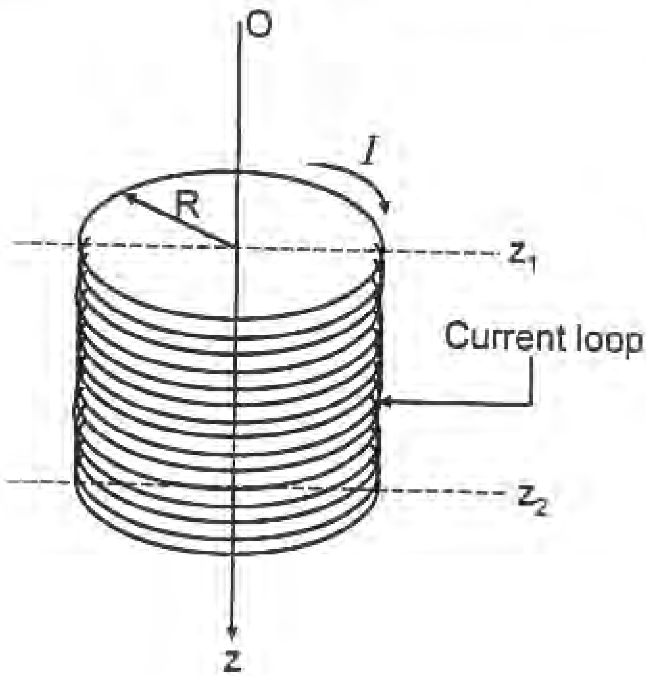
Consider a conducting rod of mass m on a tilted pair of rails in the presence of gravity g and a constant and homogeneous vertical magnetic field B (see sketch below). The angle of inclination is α. The distance between the rails is l. The rails are connected with an inductance L. The resistance of the rod and the rails and the friction between the rod and the rails are negligible. Initially the rod is at rest and there is no current in the loop ($v(0) = 0$, $I(0) = 0$). Describe the motion of the rod, i.e., obtain $v(t)$.



[I-8] [10]

Find the magnetic field at point-O (the origin of z-axis) on the axis of a tightly wound solenoid. The solenoid consists of N turns *per unit length* wrapped around a cylindrical tube of radius R and carrying current I. The top and bottom of the solenoid are at a distance z_1 and z_2 from point-O, respectively. A schematic of the solenoid is shown below. Note that you must carry out the integral.

(Hint; as the first step, you may start by first deriving B-field expression for a single current-loop.)



[I-9] [10]

An electromagnetic wave $\vec{E} = E_o\ \hat{y}\exp(i(kz - \omega t))$ is incident on an atom of polarizability α located at the origin (0, 0, 0) of a cartesian coordinate system.

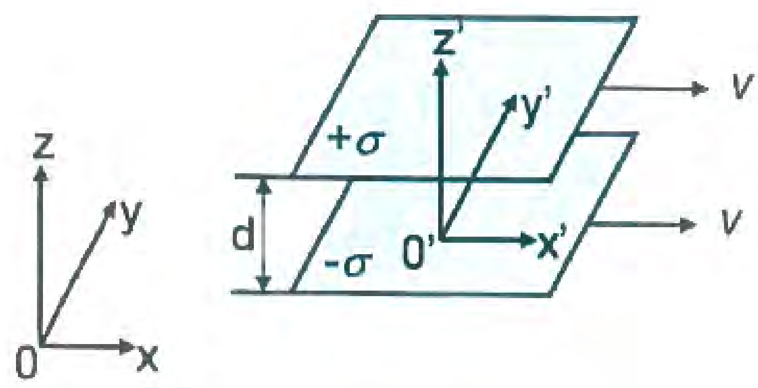
(a) Find the electric and magnetic fields of the wave radiated by the atom at large distance D on the y-axis and on the x-axis.
(b) Find the total time-averaged power radiated by the polarized atom.

Treat the atom as a dipole.

[I-10] [10]

Two large (non-conducting) parallel plates separated by a distance d and oriented as shown in the figure below, move together along the x-axis with a velocity v. The upper and lower plates have uniform charge density of $+\sigma$ and $-\sigma$, respectively, in the rest-frame of the plates.

(a) Find the magnitude and direction of the electric and magnetic fields between the plates.
(b) Now, if the parallel plates are tilted such that their surface normal is oriented along the direction of $\hat{y} + \hat{z}$. Find the magnitude and direction of the electric and magnetic fields between the plates.



Solutions Part I

I-1

A particle of mass m is bound by a linear potential U=kr.

(a) For what energy and angular momentum will the orbit be a circle of radius r about the origin?
(b) What is the frequency of this circular motion?
(c) If the particle is slightly disturbed from this circular motion, what will be the frequency of small oscillations?

solution:

The force acting on the particle is $F = -\frac{dU}{dr}\hat{r} = -k\hat{r}$.

(a) If the particle moves in a circle of radius r, $m\omega^2 r = k$, so $\omega^2 = k/mr$. The energy of the particle is then $E = kr + \frac{mv^2}{2} = kr + \frac{m\omega^2 r^2}{2} = \frac{3kr}{2}$, and it's angular momentum about the origin is $L = m\omega r^2 = mr^2\sqrt{k/mr} = \sqrt{mkr^3}$.

(b) The angular frequency of circular motion is $\omega = \sqrt{\frac{k}{mr}}$.

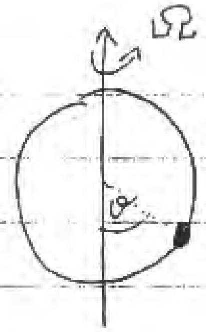
(c) The effective radial potential is $U_{eff} = kr + \frac{L^2}{2mr^2}$. The radius r_0 of stationary circular motion is given by $\left(\frac{dU_{eff}}{dr}\right)_{r=r_0} = k - \frac{L^2}{mr_0^3} = 0$, yielding $r_0 = \left(\frac{L^2}{mk}\right)^{1/3}$.

The frequency of small oscillations about r0 is given by,

$\omega_r = \sqrt{\frac{k_{eff}}{m}}$, with $k_{eff} = \left(\frac{d^2U_{eff}}{dr^2}\right)_{r=r_0} = 3L^2/mr^4|_{r=r_0} = 3k\left(\frac{mk}{L^2}\right)^{1/3}$.

$\omega_r = \sqrt{\frac{3k}{m}\left(\frac{mk}{L^2}\right)^{1/3}} = \sqrt{\frac{3k}{mr_0}} = \omega_0\sqrt{3}$, where ω_0 is the frequency of circular motion.

-2



a) $L = \frac{m}{2}\left(a^2\sin^2\theta\,\Omega^2 + a^2\dot{\theta}^2\right) + mga\cos\theta$

θ is the only generalized coordinate

b) $\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = \frac{\partial L}{\partial \theta}$

$ma^2\ddot{\theta} = \Omega^2 m a^2 \sin\theta\cos\theta - mga\sin\theta$

$$\boxed{\ddot{\theta} = \Omega^2 \sin\theta\cos\theta - \frac{g}{a}\sin\theta}$$

c) stationary angle θ: $\theta = \text{const.}$ $\dot{\theta} = 0$ $\ddot{\theta} = 0$

$0 = \Omega^2\sin\theta\cos\theta - \frac{g}{a}\sin\theta$

$0 = \left(\Omega^2\cos\theta - \frac{g}{a}\right)\sin\theta$ $\Omega_0^2 = \frac{g}{a}$

$0 = \left(\Omega^2\cos\theta - \Omega_0^2\right)\sin\theta$

i) $\sin\theta = 0 \Rightarrow$ $\boxed{\theta_1 = 0}$ $\boxed{\theta_2 = \pi}$

ii) $\Omega^2\cos\theta - \Omega_0^2 = 0 \Rightarrow$ $\cos\theta_3 = \frac{\Omega_0^2}{\Omega^2}$ $(\Omega > \Omega_0)$

$$\boxed{\theta_3 = \cos^{-1}\left(\frac{\Omega_0^2}{\Omega^2}\right)}$$

d) expand equation of motion about respective equilibrium positions up to linear order in $(\vartheta-\vartheta^*)$

where $\vartheta^* = \vartheta_1, \vartheta_2,$ or ϑ_3

$$\ddot{\vartheta} = \Omega^2 \sin\vartheta \cos\vartheta - \Omega_0^2 \sin\vartheta \approx$$

$$\approx \underbrace{\Omega^2 \sin\vartheta^* \cos\vartheta^* - \Omega_0^2 \sin\vartheta^*}_{=0 \text{ since } \vartheta^* \text{ is eq.}} +$$

$$+\left[\Omega^2 (\cos^2\vartheta^* - \sin^2\vartheta^*) - \Omega_0^2 \cos\vartheta^*\right](\vartheta-\vartheta^*) + \dots$$

$$\boxed{\ddot{\vartheta} \approx \left[\Omega^2 (2\cos^2\vartheta^* - 1) - \Omega_0^2 \cos\vartheta^*\right](\vartheta-\vartheta^*)} \equiv -\omega^2(\vartheta-\vartheta^*)$$

stability: $\omega^2 > 0$

$\boxed{\text{for } \vartheta^* = \vartheta_1 = 0:}$ $\quad \Omega^2 - \Omega_0^2 < 0 \quad$ if $\Omega^2 < \Omega_0^2$

i.e. $\vartheta_1 = 0$ stable if $\Omega < \sqrt{\frac{g}{a}}$ and $\omega = \sqrt{\Omega_0^2 - \Omega^2}$

$\boxed{\text{for } \vartheta^* = \vartheta_2 = \pi:}$ $\quad \Omega^2 + \Omega_0^2$ is always > 0

never stable

$\boxed{\text{for } \vartheta^* = \vartheta_3 = \cos^{-1}\left(\frac{\Omega_0^2}{\Omega^2}\right):}$

$$\Omega^2\left(2\frac{\Omega_0^4}{\Omega^4} - 1\right) - \Omega_0^2 \frac{\Omega_0^2}{\Omega^2} = \Omega^2\left(\frac{\Omega_0^4}{\Omega^4} - 1\right) < 0 \quad \text{if } \Omega^2 > \Omega_0^2$$

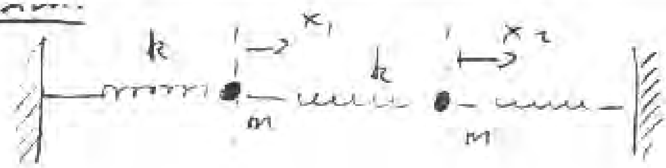
i.e. ϑ_3 is stable if $\Omega > \sqrt{\frac{g}{a}}$ and $\omega = \Omega\sqrt{1 - \frac{\Omega_0}{\Omega}}$

Summary of "phase diagram": $\left(\Omega_0 \equiv \sqrt{\frac{g}{a}}\right)$

$\Omega < \sqrt{g/a}$: $\vartheta_1 = 0$ is stable and $\quad \omega = \sqrt{\Omega_0^2 - \Omega^2}$

$\Omega > \sqrt{g/a}$: $\vartheta_3 = \cos^{-1}\left(\frac{\Omega_0^2}{\Omega^2}\right)$ is stable and $\quad \omega = \Omega\sqrt{1 - \frac{\Omega_0^4}{\Omega^4}}$

I-3.)



$$x_1(0) = D \qquad x_2(0) = 0$$
$$\dot{x}_1(0) = 0 \qquad \dot{x}_2(0) = 0$$

$$m\ddot{x}_1 = -kx_1 - k(x_1 - x_2) = -2kx_1 + kx_2$$
$$m\ddot{x}_2 = -kx_2 - k(x_2 - x_1) = kx_1 - 2kx_2$$

$\bar{x} = \bar{a}\, e^{\pm i\omega t}$
$$-\omega^2 \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} -2\,k/m & +k/m \\ k/m & -2\,k/m \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

$$\begin{pmatrix} \omega^2 - 2\frac{k}{m} & \frac{k}{m} \\ \frac{k}{m} & \omega^2 - \frac{2k}{m} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left(\omega^2 - \frac{2k}{m}\right)^2 - \left(\frac{k}{m}\right)^2 = 0$$

$$\omega^2 - \frac{2k}{m} = \pm\frac{k}{m} \quad \Rightarrow \quad \omega_1^2 = \frac{3k}{m}$$
$$\omega_2^2 = \frac{k}{m}$$

normal modes:

ω_1: $\quad \frac{k}{m} a_1 + \frac{k}{m} a_2 = 0 \quad \rightarrow \quad \bar{a}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \qquad \omega_1 = \frac{3k}{m}$

ω_2: $\quad -\frac{k}{m} a_1 + \frac{k}{m} a_2 = 0 \quad \rightarrow \quad \bar{a}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \omega_2 = \frac{k}{m}$

I-3.) continued.

Complete solution of the problem with given initial values:
(non-degenerate eigenvalues)

$$\bar{x}(t) = \left[A_1 \cos(\omega_1 t) + B_1 \sin(\omega_1 t)\right]\bar{a}_1 + \left[A_2 \cos(\omega_2 t) + B_2 \sin(\omega_2 t)\right]\bar{a}_2$$

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \left[A_1 \cos(\omega_1 t) + B_1 \sin(\omega_1 t)\right]\begin{pmatrix} 1 \\ -1 \end{pmatrix} + \left[A_2 \cos(\omega_2 t) + B_2 \sin(\omega_2 t)\right]\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$(1) \quad \begin{cases} x_1(t) = A_1 \cos(\omega_1 t) + B_1 \sin(\omega_1 t) + A_2 \cos(\omega_2 t) + B_2 \sin(\omega_2 t) \\ x_2(t) = -A_1 \cos(\omega_1 t) - B_1 \sin(\omega_1 t) + A_2 \cos(\omega_2 t) + B_2 \sin(\omega_2 t) \end{cases}$$

$$(2) \quad \begin{cases} \dot{x}_1(t) = -\omega_1 A_1 \sin(\omega_1 t) + \omega_1 B_1 \cos(\omega_1 t) - \omega_2 A_2 \sin(\omega_2 t) + \omega_2 B_2 \cos(\omega_2 t) \\ \dot{x}_2(t) = \omega_1 A_1 \sin(\omega_1 t) - \omega_1 B \cos(\omega_1 t) - \omega_2 A_2 \sin(\omega_2 t) + \omega_2 B_2 \cos(\omega_2 t) \end{cases}$$

$\underline{t=0}$, $x_1(0) = D$ $\quad x_2(0) = \emptyset$ $\quad$ from (1):

$$\left.\begin{aligned} D &= A_1 + A_2 \\ 0 &= -A_1 + A_2 \end{aligned}\right\} \Rightarrow \quad A_1 = A_2 = \frac{D}{2}$$

$\dot{x}_1(0) = \dot{x}_2(0) = 0$ $\quad$ from (2):

$$\left.\begin{aligned} \omega_1 B_1 + \omega_2 B_2 &= 0 \\ -\omega_1 B_1 + \omega_2 B_2 &= 0 \end{aligned}\right\} \Rightarrow \quad B_1 = B_2 = 0$$

Full solution:

$$x_1(t) = \frac{D}{2}\left(\cos(\omega_1 t) + \cos(\omega_2 t)\right) = D\cos\left(\frac{\omega_1+\omega_2}{2} t\right)\cdot\cos\left(\frac{\omega_1-\omega_2}{2} t\right)$$

$$x_2(t) = \frac{D}{2}\left(-\cos(\omega_1 t) + \cos(\omega_2 t)\right) = D\sin\left(\frac{\omega_1+\omega_2}{2} t\right)\sin\left(\frac{\omega_1-\omega_2}{2} t\right)$$

$\omega_1 = \sqrt{\frac{3k}{m}} \quad , \omega_2 = \sqrt{\frac{k}{m}}$

$$\boxed{\begin{aligned} x_1(t) &= D \cos\left(\sqrt{\frac{k}{m}}\,\frac{\sqrt{3}+1}{2}\,t\right) \cos\left(\sqrt{\frac{k}{m}}\,\frac{\sqrt{3}-1}{2}\,t\right) \\ x_2(t) &= D \sin\left(\sqrt{\frac{k}{m}}\,\frac{\sqrt{3}+1}{2}\,t\right) \sin\left(\sqrt{\frac{k}{m}}\,\frac{\sqrt{3}-1}{2}\,t\right) \end{aligned}}$$

I-4 (a) gravitation field $\vec{g} = -GM/r^2$

Surface integral $\oint \vec{g} \cdot d\vec{A} = -4\pi G (M_{enclosed})$ #

(b) for $r < R$, $M_{enclosed} = \rho \cdot (\frac{4}{3}\pi r^3)$,

$\rho = M / \frac{4}{3}\pi R^3$

$\therefore M_{enclosed} = M(r^3/R^3)$ #

from Gauss law: $g \cdot \cancel{4\pi} r^2 = -\cancel{4\pi} G \left[M (r^3/R^3) \right]$

$\therefore g(r) = -GM(r/R^3)$ #

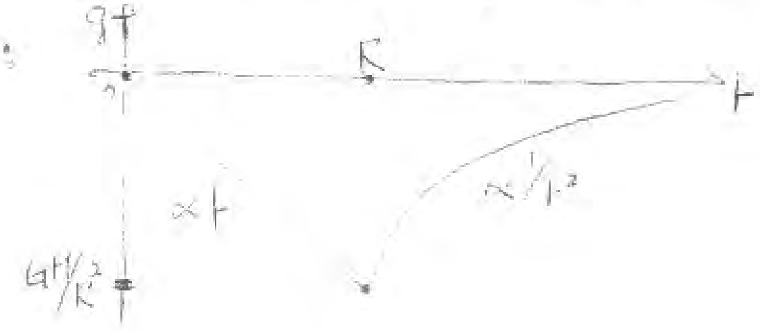
for $r > R$, $M_{enclosed} = M$

from Gauss law $g \, 4\pi r^2 = -4\pi G \left[M \right]$

$\therefore g(r) = -GM(1/r^2)$ #

for $r = R$, $g(r=R) = -GM(1/R^2)$ #

plot:



I-4 (c) for $r < R$, $U(r) = G\frac{M}{R^3}\left(\frac{r^2}{2}\right) + C$ #

for $r > R$, $U(r) = -GM\left(\frac{1}{r}\right)$ #

$(U(r\to\infty) = 0)$

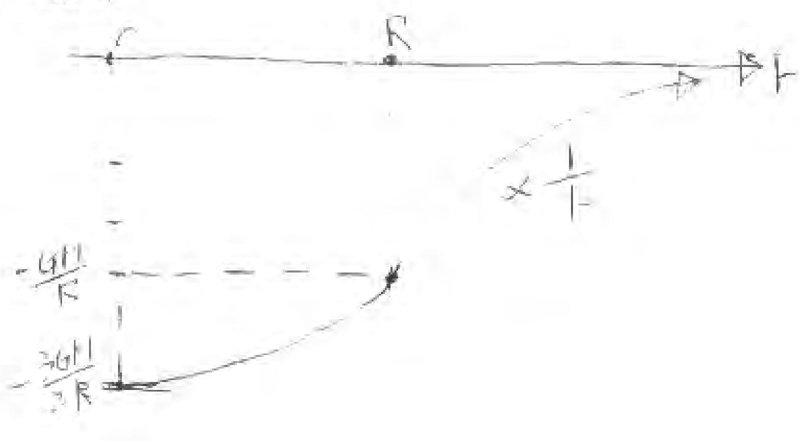
Apply continuity condition at $r = R$

$$G\frac{M}{R^3}\left(\frac{R^2}{2}\right) + C = -GM\frac{1}{R}$$

$$C = \frac{GM}{R}\left(\frac{-3}{2}\right) \#$$

$$\therefore U(r) = \begin{cases} GM\left[\frac{r^2}{2R^3} - \frac{3}{2R}\right] & r \le R \\ -GM\left[\frac{1}{r}\right] & r \ge R \quad \# \\ -\frac{GM}{R}\frac{3}{2} & r = 0 \end{cases}$$

(d) $U(r)$

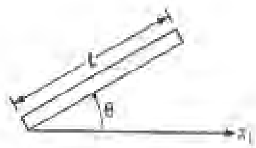


Part I – Mechanics

I-5 Special Relativity [10]

A stick of lengths l is fixed at an angle θ from its x_I-axis in its own rest system K.

What is the lengths and orientation of the stick as measured by an observer moving along x_I with relativistic speed v?



I-5 Solution

For an observer in motion relative to an object, the dimensions of objects are contracted by a factor of $\sqrt{1-v^2/c^2}$ in the direction of motion. Thus, the x_1' component of the stick will be

$$\ell \cos\theta \sqrt{1-v^2/c^2}$$

while the perpendicular component will be unchanged:

$$\ell \sin\theta$$

So, to the observer in K', the length and orientation of the stick are

$$\ell' = \ell\left[\sin^2\theta + \left(1-v^2/c^2\right)\cos^2\theta\right]^{1/2}$$

$$\theta' = \tan^{-1}\left[\frac{\sin\theta}{\cos\theta\sqrt{1-v^2/c^2}}\right]$$

or

$$\boxed{\begin{aligned} \ell' &= \ell\left[\sin^2\theta + \frac{\cos^2\theta}{\gamma^2}\right]^{1/2} \\ \tan\theta' &= \gamma \tan\theta \end{aligned}}$$

I-6

Suppose that the region z>0 in three-dimensional space is filled with a linear dielectric material characterized by a dielectric constant ϵ_1, while the region z<0 has a dielectric material ϵ_2. Fix a charge −q at (x,y,z) = (0,0,a) and a charge +q at (0,0,-a). What is the force one must exert on the negative charge to keep it at rest?

solution:

Consider first the case where a point charge q_1 is placed at (0,0,a). The method of images requires image charges q_1' at (0,0,-a) and q_1'' at (0,0,a). Then the potential (in Gaussian units) at a point (x,y,z) is given by $\phi_1 = \frac{q_1}{\epsilon_1 r_1} + \frac{q_1'}{\epsilon_1 r_2}$, $(z \geq 0)$, $\phi_2 = \frac{q_1''}{\epsilon_2 r_1}$, $(z < 0)$,

where, $r_1 = \sqrt{x^2 + y^2 + (z-a)^2}$, $r_2 = \sqrt{x^2 + y^2 + (z+a)^2}$

Applying the boundary conditions at (x,y,0):

$$\phi_1 = \phi_2, \qquad \epsilon_1 \frac{\partial \phi_1}{\partial z} = \epsilon_2 \frac{\partial \phi_2}{\partial z}$$

we obtain,

$$q_1' = q_1'' = \frac{(\epsilon_1 - \epsilon_2)}{\epsilon_1(\epsilon_1 + \epsilon_2)} q_1$$

Similarly, for a point charge q_2 placed at (0,0,-a) inside the dielectric ϵ_2, its image charges will be q_2' at (0,0,a) and q_2'' at (0,0,-a) with magnitudes $q_2' = q_2'' = \frac{\epsilon_2 - \epsilon_1}{\epsilon_2(\epsilon_1 + \epsilon_2)} q_2$.

When both q_1 and q_2 exist, the force on q_1 will be the resultant due to q_2,q_1', and q_1''. It follows that, $F = \frac{q_1 q_1'}{4a^2 \epsilon_1} + \frac{q_1 q_2}{4a^2 \epsilon_2} + \frac{q_1 q_1''}{4a^2 \epsilon_2} = \frac{\epsilon_1 - \epsilon_2}{\epsilon_1(\epsilon_1 + \epsilon_2)} q_1^2/4a^2 \;\; + q_1 q_2 / 2(\epsilon_1 + \epsilon_2) a^2$.

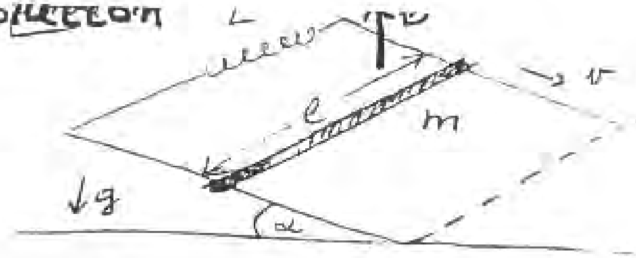
For the current problem, $q_1 = -q$ and $q_2 = +q$, on has

$$F = \frac{\epsilon_1 - \epsilon_2}{\epsilon_1(\epsilon_1 + \epsilon_2)} \frac{q^2}{4a^2} - \frac{q^2}{2(\epsilon_1 + \epsilon_2) a^2} = -\frac{q^2}{4\epsilon_1 a^2}.$$

Hence, a force of −F is required to keep −q at rest.

SOLUTION

1-7



$$\boxed{\begin{array}{l} v(0)=0 \\ I(0)=0 \end{array}}$$ initial c.

(1) $m\dot{v}(t) = mg\sin\alpha - I(t)\ell B\cos\alpha$ (Newton's II.)

(2) $v(t)\ell B\cos\alpha - L\dot{I}(t) = 0$ (voltages in closed loop)

$$m\ddot{v} = -\dot{I}\ell B\cos\alpha = -\frac{v\ell B\cos\alpha}{L}\cdot \ell B\cos\alpha$$

$$m\ddot{v} = -\frac{\ell^2B^2\cos\alpha}{L}v(t)$$

$$\ddot{v} = -\frac{\ell^2B^2\cos\alpha}{Lm}v(t) \implies \boxed{\omega = \frac{\ell B\cos\alpha}{\sqrt{mL}}}$$

(harmonic oscillation)

(3) $\begin{cases} v(t) = A\cos(\omega t) + B\sin(\omega t) \\ \dot{v}(t) = -\omega A\sin(\omega t) + \omega B\cos(\omega t) \end{cases}$

need $v(0)$ and $\dot{v}(0)$

$v(0) = 0$

and from $I(0)=0$ & (1): $m\dot{v}(0) = mg\sin\alpha \implies \dot{v}(0) = g\sin\alpha$

$t=0$; from (3): $0 = A$

$$g\sin\alpha = \omega B \implies B = \frac{g\sin\alpha}{\omega} = \frac{\sqrt{mL}\cdot g\sin\alpha}{\ell B\cos\alpha}$$

$$\boxed{v(t) = \frac{g\sin\alpha\sqrt{mL}}{\ell B\cos\alpha}\cdot\sin\left(\frac{\ell B\cos\alpha}{\sqrt{mL}}t\right)}$$

I-8

For a single current loop, located at position $z = z_0$, the B-field at point 0

$$B_z = \frac{\mu_0 I}{2} \frac{R^2}{(R^2 + z_0^2)^{3/2}}$$

#

$$B_x = B_y = 0 \quad (\text{by symmetry})$$

The solenoid has N turns per unit length,

$$dI = (dz) N \cdot I$$

$$dB_z = \frac{\mu_0}{2}(dI) \frac{R^2}{(R^2 + z^2)^{3/2}}$$

$$B_z = \frac{\mu_0 N I}{2} \int_{z_1}^{z_2} \frac{R^2 \, dz}{(R^2 + z^2)^{3/2}}$$

$$B_z = \frac{\mu_0 N I}{2} \left. \frac{R^2 z}{R^2 (R^2 + z^2)^{1/2}} \right|_{z_1}^{z_2}$$

#

Solution:

[I-9]

induced dipole moment $\vec{p} = \alpha \vec{E}$

a) Scattered fields

$$\vec{E}_{sc} = \frac{1}{4\pi\varepsilon_0} k^2 \frac{e^{-ikr}}{r} [(\vec{n} \times \vec{p}) \times \vec{n}]$$

$$\vec{H}_{sc} = \vec{n} \times \vec{E}_{sc} / Z_0 \qquad \text{or} \quad \vec{E} = \frac{(\vec{p} \times \vec{n}) \times \vec{n}}{4\pi\varepsilon_0 c^2 R}$$

$$\vec{p} = \alpha E_0 e^{i(kz - \omega t)} \vec{e}_y \qquad \vec{B} = \frac{\vec{p} \times \vec{n}}{4\pi\varepsilon_0 c^3 R}$$

$\hat{n}$: unit vector between scatter & observer

$$\vec{p} = \begin{pmatrix} 0 \\ p \\ 0 \end{pmatrix}$$

(i) on y-axis

$\vec{n} = \vec{e}_y \quad r = D$

$$(\vec{n} \times \vec{p}) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ p \\ 0 \end{pmatrix} = 0$$

$\therefore (\vec{n} \times \vec{p}) \times \vec{n} = 0$

$\therefore \vec{E}_{sc} = 0, \ \vec{H}_{sc} = 0$ on y-axis

(ii) on x-axis $\vec{n} = \vec{e}_x \quad r = D$

$$(\vec{n} \times \vec{p}) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ p \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ p \end{pmatrix}$$

$$(\vec{n} \times \vec{p}) \times \vec{n} = \begin{pmatrix} 0 \\ 0 \\ p \end{pmatrix} \times \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ p \\ 0 \end{pmatrix} \qquad \vec{E}_{sc} = \frac{1}{4\pi\varepsilon_0} k^2 \frac{e^{-ikD}}{D} \alpha E_0 e^{i(kD - \omega t)} \vec{e}_y$$

$$\vec{H}_{sc} = \vec{n} \times \vec{E}_{sc} / Z_0 \qquad \vec{H}_{sc} = H_{sc} \vec{e}_z$$

b) $S = \frac{1}{2} \frac{E^2}{Z_0}$ $\qquad$ power $P = \int dt \int d\Omega \, S = \dfrac{\alpha^2 \omega^4 E_0^2}{12\pi\varepsilon_0 c^3}$

[I-10]

let EM field be $\vec{E}', \vec{B}'$ in frame-S': (x', y')

let EM field be $\vec{E}, \vec{B}$ in frame-S: (x, y)

let $\beta \stackrel{\text{def}}{=} v/c$, $\gamma \stackrel{\text{def}}{=} 1/\sqrt{1-\beta^2}$

we have:

$$\begin{cases} E_x = E_x' \\ E_y = \gamma (E_y' + \beta c B_z') \\ E_z = \gamma (E_z - \beta c B_y) \end{cases}$$

$$\begin{cases} B_x = B_x' \\ B_y = \gamma (B_y - \frac{\beta}{c} E_z') \\ B_z = \gamma (B_z' + \frac{\beta}{c} E_y') \end{cases} \#$$

(b) Now in the rest frame-S':

$$B_x' = B_y' = B_z' = 0$$

$$E_x' = E_y' = 0$$

$$E_z' = \sigma/\varepsilon_0$$

$$\therefore \begin{cases} E_x = 0 \quad E_y = 0 \quad E_z = \gamma\left(\frac{\sigma}{\varepsilon_0}\right) \# \\ B_x = 0 \quad B_y = -\frac{\gamma\beta}{c}\left(\frac{\sigma}{\varepsilon_0}\right), \quad B_z = 0 \# \end{cases}$$

(b) in the rest-frame →

$$B_x' = B_y' = B_z' = 0$$

$$E_x' = 0$$

$$E_y' = E_z' = \frac{\sigma_0}{\varepsilon_0\sqrt{2}} \#$$

$$\therefore \quad E_x = 0, \qquad E_y = \gamma\left(\frac{\sigma_0}{\varepsilon_0\sqrt{2}}\right), \qquad E_z = \gamma\left(\frac{\sigma_0}{\varepsilon_0\sqrt{2}}\right) \#$$

$$B_x = 0, \qquad B_y = -\frac{\beta\gamma}{c}\left(\frac{\sigma_0}{\varepsilon_0\sqrt{2}}\right), \qquad B_z = \frac{\beta\gamma}{c}\left(\frac{\sigma_0}{\varepsilon_0\sqrt{2}}\right) \#$$

solutions

Physics PhD Qualifying Examination
Part II – Friday, January 17, 2014

Name:______________________________
(please print)
Identification Number:________

STUDENT: insert a check mark in the left boxes to designate the problem numbers that you are handing in for grading.
PROCTOR: check off the right hand boxes corresponding to the problems received from each student. Initial in the right hand box.

	1	
	2	
	3	
	4	
	5	
	6	
	7	
	8	
	9	
	10	

Student's initials
problems handed in:
Proctor's initials

INSTRUCTIONS FOR SUBMITTING ANSWER SHEETS

1. DO NOT PUT YOUR NAME ON ANY ANSWER SHEET. EXAMS WILL BE COLLATED AND GRADED BY THE ID NUMBER ABOVE.
2. Use at least one separate preprinted answer sheet for each problem. Write on only one side of each answer sheet.
3. Write your **identification number** listed above, in the appropriate box on the preprinted sheets.
4. Write the **problem number** in the appropriate box of each preprinted answer sheet. If you use more than one page for an answer, then number the answer sheets with both problem number and page (e.g. Problem 9 – Page 1 of 3).
5. Staple together all the pages pertaining to a given problem. Use a paper clip to group together all problems that you are handing in.
6. A passing distribution for the individual components will normally include at least four passed problems (from problems 1-6) for Quantum Physics and two problems (from problems 7-10) for Thermodynamics and Statistical Mechanics.
7. **YOU MUST SHOW ALL YOUR WORK.**

[II-1] **[10]**

A quantum mechanical system is known to possess only two energy eigenstates denoted $|1\rangle$ and $|2\rangle$. The system also includes three other observables (besides the energy), known as P, Q, and R. The states denoted $|1\rangle$ and $|2\rangle$ are normalized but they are not necessarily eigenstates of P, Q, or R. A series of experiments are performed yielding,

Experiment 1: $\langle 1|P|1\rangle = 1/2$ and $\langle 1|P^2|1\rangle = 1/4$

Experiment 2: $\langle 1|Q|1\rangle = 1/2$ and $\langle 1|Q^2|1\rangle = 1/6$

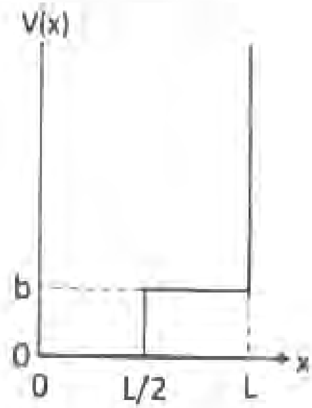
Experiment 3: $\langle 1|R|1\rangle = 1$, $\langle 1|R^2|1\rangle = 5/4$, and $\langle 1|R^3|1\rangle = 7/4$

(a) One set of experimental data is erroneous and yields unphysical results. Which dataset is it?

(b) Determine as many eigenvalues of P, Q, and R as possible from the information given.

[II-2] **[10]**

A particle moves in a one-dimensional box with small potential hump in half of the well, as shown in the figure.



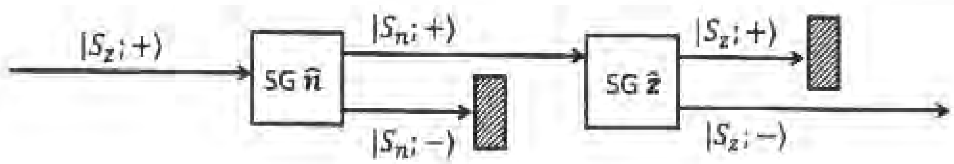
$$V(x) = \begin{cases} \infty \ (x < 0, x > L) \\ 0 \ (x > 0, x < \frac{L}{2}) \\ b \ (x > \frac{L}{2}, x < L) \end{cases}$$

Treating the potential hump as a perturbation to the "regular" infinite potential well problem, with $V(x) = \infty \ (x < 0, x > L)$ and $V(x) = 0 \ (0 < x < L)$, determine the first order energy of the ground state.

[II-3] **[10]**

A beam of identically prepared spin ½ atoms with $S_z = +\hbar/2$ orientation and with *unit intensity* goes through a series of Stern-Gerlach-type (SG) measurements (selective filtering) as follows:

- The first measurement accepts $S_n = +\hbar/2$ atoms (and rejects $S_n = -\hbar/2$ ones), where S_n is the eigenvalue of the spin operator along the direction $\hat{n} = (\cos\varphi\sin\vartheta, \sin\varphi\sin\vartheta, \cos\vartheta)$, $\boldsymbol{S}\cdot\hat{\boldsymbol{n}}$. ($\vartheta$ and φ are the polar and azimuthal and angles, respectively).
- The second measurement accepts $S_z = -\hbar/2$ atoms (and rejects $S_z = +\hbar/2$ ones).



What is the intensity of the final $S_z = -\hbar/2$ beam? (You must express your answer in terms of ϑ and φ.)

[II-4] **[10]**

(a) Consider a potential of the form $V(r) = V_o \dfrac{e^{-ar}}{r}$.

(i) Calculate the differential scattering cross section using Born approximation.
(ii) Calculate the total scattering cross section.

(b) Using the results of (a), calculate (*i*) the differential and (*ii*) the total scattering cross section of Coulomb potential $V_C(r) = \dfrac{Z_1 Z_2 e^2}{r}$

[II-5] **[10]**

(a) Write down uncertainty relationship of space and momentum;
(b) Use the uncertainty relationship to estimate the ground-state energy of a particle, mass m, in a one-dimensional infinite potential well of width L;
(c) Use the uncertainty relationship to estimate the ground-state energy of a one-dimensional simple harmonic oscillator of mass m and oscillating angular-frequency ω.

[II-6] **[10]**

$\Psi(x, t)$ is a solution of the time-dependent Schroedinger equation

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\psi + V(\vec{x})\psi.$$

Show that $\rho(\vec{x},t) = |\psi(x,t)|^2$ and $\vec{j}(\vec{x},t) = \frac{i\hbar}{2m}\left(\psi\vec{\nabla}\psi^* - \psi^*\vec{\nabla}\psi\right)$ satisfy the continuity equation

$$\frac{\partial\rho}{\partial t} + \vec{\nabla}\bullet\vec{j} = 0.$$

[II-7] [10]

A gas is initially confined to one-half of a thermally isolated container. The other half is empty. The gas is suddenly permitted to expand to fill the entire chamber ((Gay-Lussac — Joule) "*free expansion*"). Assuming the initial temperature of the gas in the half-container is T_i find the temperature T_f after the expansion for the following two cases:

(a) Equation of state: $pV=nRT$
(b) Equation of state: $b\,(p+(a/V^2))=nRT$

a and b are positive constants. p, V and T denote pressure, volume and temperature, respectively. R is the universal gas constant and n is the mol number.

[II-8] [2,4,4]

The Helmholtz free energy of a dilute plasma gas consisting of N electrons confined to a volume V at temperature T is given by

$$F(T,V,N)=F^{ideal}(T,V,N)-\frac{2}{3}Ne^2\left(\frac{4\pi Ne^2}{Vk_BT}\right)^{1/2},$$

where e is the electron charge, k_B is Boltzmann's constant, and $F^{ideal}(T,V,N)$ is the Helmholtz free energy of a mono-atomic ideal gas.

(a) Obtain the equation of state of the above plasma gas.
(b) Obtain the internal energy of the system $E(T,V,N)$.
(c) Obtain the constant-volume heat capacity C_V.

Note: your answers should be fully explicit in terms of the variables T, V, and N. To that end, you are expected to remember and use the equation of state, internal energy, heat capacity of the ideal gas in obtaining your final results.

[II-9] **[10]**

In the Einstein model for a three-dimensional solid of N atoms, the system is treated as an ensemble of $3N$ *distinguishable* and *independent* quantum harmonic oscillators with *identical* frequency ω_o (three oscillators for each atoms).

(a) Find the specific heat of this simple model for solids (the Einstein crystal). You must express your answer in terms of N, k_B (Boltzmann's constant), and the ratio θ_E / T, where $\theta_E = \hbar\omega_o / k_B$ is the characteristic Einstein temperature.

(b) Obtain the high-temperature behavior of the specific heat. How does it compare to the classical behavior?

(c) Obtain the low-temperature behavior of the specific heat. What can you say about its limiting value as $T \to 0$?

[II-10] **[10]**

Consider a *three-dimensional* extreme-relativistic ($\varepsilon = cp$) free electron gas confined to a volume $V = L^3$. The number of electrons is N.

Obtain P_o, the pressure of the system at $T = 0$. You must express your answer in terms of the electron (number) density N / V.

Solutions Part II

II-1

A quantum mechanical system is known to possess only two energy eigenstates denoted $|1\rangle$ and $|2\rangle$. The system also includes three other observables (besides the energy), known as P,Q, and R. The states denoted $|1\rangle$ and $|2\rangle$ are normalized but they are not necessarily eigenstates of P,Q, or R. A series of experiments are performed yielding:

Experiment 1: $\langle 1|P|1\rangle = 1/2$ and $\langle 1|P^2|1\rangle = 1/4$

Experiment 2: $\langle 1|Q|1\rangle = 1/2$ and $\langle 1|Q^2|1\rangle = 1/6$

Experiment 3: $\langle 1|R|1\rangle = 1$, $\langle 1|R^2|1\rangle = 5/4$, and $\langle 1|R^3|1\rangle = 7/4$

(a) One set of experimental data is erroneous and yields unphysical results. Which dataset is it?
(b) Determine as many eigenvalues of P,Q, and R as possible from the information given.

Solution:

(a) The results from experiment 2 are unphysical. Writing Q as a Hermitian operator, $Q = \begin{pmatrix} \alpha & \beta \\ \beta^* & \gamma \end{pmatrix}$, the results of the experiment can bet written as:

$\langle 1|Q|1\rangle = (1 \quad 0)\begin{pmatrix} \alpha & \beta \\ \beta^* & \gamma \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \alpha = \frac{1}{2}$

and,

$$\langle 1|Q^2|1\rangle = (1 \quad 0)\begin{pmatrix} \alpha & \beta \\ \beta^* & \gamma \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ \beta^* & \gamma \end{pmatrix} = (1 \quad 0)\begin{pmatrix} \alpha & \beta \\ \beta^* & \gamma \end{pmatrix}\begin{pmatrix} \alpha \\ \beta^* \end{pmatrix} = (1 \quad 0)\begin{pmatrix} \alpha^2 + \beta\beta^* \\ \beta^*\alpha + \gamma\beta^* \end{pmatrix} = \alpha^2 + \beta\beta^*$$
$$= \frac{1}{6}$$

The result $\langle 1|Q|1\rangle = \frac{1}{2}$ requires $= \frac{1}{2}$. However, from $\langle 1|Q^2|1\rangle = \frac{1}{6}$ we see that no β can yield the "observed" results, as this requires $\beta\beta^* = -\frac{1}{12}$

(b)

Operator P:

(i) $\langle 1|P|1\rangle = 1/2 \rightarrow \alpha = 1/2$

(ii) $\langle 1|P^2|1\rangle = 1/4 \rightarrow \alpha^2 + \beta\beta^* = \frac{1}{4}, \ \beta = 0$

Therefore, $P = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \gamma \end{pmatrix}$, where γ cannot be determined. It is already diagonal, and hence:

$$\lambda_1 = \frac{1}{2}, \qquad \lambda_2 = \gamma = unknown$$

Operator R:

(i) $\langle 1|R|1\rangle = 1 \rightarrow \alpha = 1$

(ii) $\langle 1|R^2|1\rangle = 5/4 \rightarrow \alpha^2 + \beta\beta^* = \frac{5}{4}, \beta\beta^* = \frac{1}{4}, \beta = \frac{1}{2}e^{i\theta_\beta}$

(iii) $\langle 1|R^3|1\rangle = (1 \quad 0)\begin{pmatrix}\alpha & \beta \\ \beta^* & \gamma\end{pmatrix}\begin{pmatrix}\alpha^2 + \beta\beta^* \\ \beta^*\alpha + \gamma\beta^*\end{pmatrix} = (1 \quad 0)\begin{pmatrix}\alpha(\alpha^2 + \beta\beta^*) + \beta(\beta^*\alpha + \gamma\beta^*) \\ \beta^*(\alpha^2 + \beta\beta^*) + \gamma(\beta^*\alpha + \gamma\beta^*)\end{pmatrix} =$

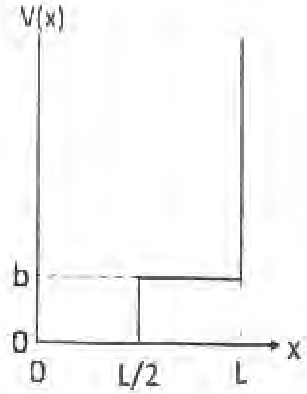
$\alpha(\alpha^2 + \beta\beta^*) + \beta(\beta^*\alpha + \gamma\beta^*) = \left(1 + \frac{1}{4}\right) + \frac{1}{4} + \frac{\gamma}{4} = \frac{7}{4}, \gamma = 1.$

Therefore, $R = \begin{pmatrix} 1 & \frac{1}{2}e^{i\theta_\beta} \\ \frac{1}{2}e^{-i\theta_\beta} & 1 \end{pmatrix}$, and $\begin{vmatrix} 1-\lambda & \frac{1}{2}e^{i\theta_\beta} \\ \frac{1}{2}e^{-i\theta_\beta} & 1-\lambda \end{vmatrix} = (1-\lambda)^2 - \frac{1}{4} = 0.$

with eigenvalues $\lambda_1 = \frac{1}{2}, \quad \lambda_2 = \frac{3}{2}.$

II-2

A particle moves in a one-dimensional box with small potential hump in half of the well, as shown in the figure.



$$V(x) = \begin{cases} \infty \ (x < 0, x > L) \\ 0 \ \ (x > 0, x < \frac{L}{2}) \\ b \ (x > \frac{L}{2}, x < L) \end{cases}$$

Treating the potential hump as a perturbation to the "regular" infinite potential well problem ($V(x) = \infty \ (x < 0, x > L); V(x) = 0 \ (0 < x < L)$), determine the first order energy of the ground state.

solution:

For the ground state of the "regular" infinite potential well we have:

$\Psi_0 = A\sin(\frac{\pi x}{L})$, with energy $E_0 = \frac{\pi^2\hbar^2}{2mL^2}$. The perturbation $H' = V'(x) = \begin{cases} 0 \ (x < \frac{L}{2}) \\ b \ (x > \frac{L}{2}) \end{cases}$

In the context of first order perturbation theory,

$$E = E_0 + \langle\Psi_0|H'|\Psi_0\rangle$$

$$= E_0 + \int_0^L V'(x)A^2\sin^2\frac{\pi x}{L}dx = E_0 + \int_0^{\frac{L}{2}} 0 * A^2\sin^2\frac{\pi x}{L}dx + \int_{\frac{L}{2}}^{L} bA^2\sin^2\frac{\pi x}{L}dx$$

$$= E_0 + b\int_{\frac{L}{2}}^{L} A^2\sin^2\frac{\pi x}{L}dx = E_0 + \frac{b}{2}$$

Therefore, $E = \frac{\pi^2\hbar^2}{2mL^2} + \frac{b}{2}$

$\boxed{II-3}$

$$\begin{pmatrix} |+\rangle \equiv |S_z;+\rangle \\ |-\rangle \equiv |S_z;-\rangle \end{pmatrix}$$

$$|S_n;+\rangle = |\vec{S}\cdot\hat{n};+\rangle = \cos(\theta/2)|+\rangle + \sin(\theta/2)\, e^{i\varphi}|-\rangle$$

$$\left(|S_n;-\rangle = |\vec{S}\cdot\hat{n};-\rangle = \sin(\theta/2)|+\rangle - \cos(\theta/2)\, e^{i\varphi}|-\rangle \right)$$

$$P_1 = |\langle S_n;+|+\rangle|^2 = |\langle +|S_n;+\rangle|^2 = \cos^2(\theta/2)$$

$$P_2 = |\langle -|S_n;+\rangle|^2 = \sin^2(\theta/2)$$

$$I_{final} = 1\cdot P_1\cdot P_2 = \cos^2(\theta/2)\sin^2(\theta/2) = \frac{1}{4}\left[2\cos(\theta/2)\sin(\theta/2)\right]^2$$

$$\boxed{= \frac{1}{4}\sin^2(\theta)}$$

(II-4)

(a) From Born Approximation:

$$f^{(1)}(\vec{k}',\vec{k}) = \frac{-1}{4\pi}\frac{2\mu}{\hbar^2}\int d\vec{r}^{\,3}\; e^{i(\vec{k}-\vec{k}')\cdot\vec{r}}\; V(\vec{r})$$

(i) $|\vec{k}-\vec{k}'| = 2k\sin\theta/2$

$|\vec{K}| = 2k\sin\theta/2$

(ii) $$f^{(1)} = -\frac{1}{4\pi}\frac{2\mu V_0}{\hbar^2}\int_0^\infty r\,dr\;\sin(|\vec{K}|r)\,\frac{e^{-\alpha r}}{r}$$

$$= -\frac{2\mu V_0}{\hbar^2}\frac{1}{\alpha^2+|\vec{K}|^2}$$

$$= \frac{-2\mu V_0}{\hbar^2}\frac{1}{[\alpha^2+4k^2\sin^2\theta/2]}$$

$$\sigma(\theta) = \frac{4\mu^2V_0^2}{\hbar^4}\frac{1}{[\alpha^2+4k^2\sin^2\theta/2]^2}$$

$$\sigma_{total} = \int d\Omega\,\sigma(\theta) = \frac{4\mu^2V_0^2}{\hbar^4}\frac{4\pi}{\alpha^2(\alpha^2+4k^2)}$$

#

(II-4) ③/2

(b) For Coulomb potential, $V = \frac{Z_1 Z_2}{r} e^2$

Set $V_0 = Z_1 Z_2 e^2$ & $\alpha = 0$

(i) $\sigma(\theta) = \frac{4\mu^2}{\hbar^4} \frac{Z_1^2 Z_2^2 e^4}{16 k^4 \sin^4 \theta/2}$

(ii) $\sigma_{total} \to \infty$

#

II-5

(a) The uncertainty relationship:

$$\Delta x \cdot \Delta p \approx \hbar/2 \quad \#$$

(b) for 1D infinite well of width L

Given: $\Delta x = L$

from uncertainty relationship: $\Delta p \approx \left(\frac{\hbar}{2}\right)\left(\frac{1}{\Delta x}\right) = \frac{\hbar}{2\Delta x} = \frac{\hbar}{2L}$ #

Total energy: $E_0 = \frac{p^2}{2m} + V = \frac{1}{2m}\left[\frac{\hbar^2}{4L^2}\right] + 0$

$$\therefore E_0 = \frac{\hbar^2}{8mL^2} \quad \#$$

Note: [illegible]

(c) total energy of a harmonic oscillator

$$E = \frac{p_x^2}{2m} + \frac{1}{2} m \omega^2 x^2 \quad \#$$

from uncertainty relationship $\Delta p_x \Delta x = \hbar/2$

$$\therefore E = \frac{\hbar^2}{8m\Delta x^2} + \frac{1}{2} m \omega^2 \Delta x^2$$

To obtain ground state energy, we minimize the whole eqn

II-5 $\quad \therefore \dfrac{-\hbar^2}{2m\,\Delta x^3} + m\omega^2 \Delta x = 0$

$$\Delta x = \sqrt{\frac{\hbar}{2m\omega}} \quad \#$$

$$E = \frac{\hbar^2}{8m\left[\frac{\hbar}{2m\omega}\right]} + \frac{1}{2}m\omega^2\left[\frac{\hbar}{2m\omega}\right]$$

Ground state energy: $E = \frac{1}{4}\hbar\omega + \frac{1}{4}\hbar\omega = \underline{\frac{1}{2}\hbar\omega}$ #

II-6

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\psi + V(\vec r)\psi \Rightarrow \frac{\partial\psi}{\partial t} - \frac{i\hbar}{2m}\Delta\psi + \frac{V(\vec r)\psi}{i\hbar} = 0$$

$$-i\hbar\frac{\partial\psi^*}{\partial t} = -\frac{\hbar^2}{2m}\Delta\psi^* + V(\vec r)\psi^* \Rightarrow \frac{\partial\psi^*}{\partial t} + \frac{i\hbar}{2m}\Delta\psi^* + \frac{V(\vec r)\psi^*}{-i\hbar} = 0$$

$$\rho(\vec r,t) = |\psi(\vec r,t)|^2 = \psi^*(\vec r,t)\,\psi^*(\vec r,t)$$

$$\vec j(\vec r,t) = \frac{i\hbar}{2m}\left(\psi\vec\nabla\psi^* - \psi^*\vec\nabla\psi\right)$$

$$\frac{\partial\rho}{\partial t} + \vec\nabla\vec j = 0$$

$$\frac{\partial\rho}{\partial t} = \frac{\partial}{\partial t}(\psi^*\psi) = \dot\psi^*\psi + \psi^*\dot\psi$$

$$\vec\nabla\vec j = \frac{i\hbar}{2m}\vec\nabla\left(\psi\vec\nabla\psi^* - \psi^*\vec\nabla\psi\right) = \frac{i\hbar}{2m}\left(\vec\nabla\psi\vec\nabla\psi^* + \psi\Delta\psi^* - \vec\nabla\psi^*\vec\nabla\psi - \psi^*\Delta\psi\right)$$

$$\vec\nabla\vec j = \frac{i\hbar}{2m}\left(\psi\Delta\psi^* - \psi^*\Delta\psi\right)$$

$$\psi\frac{\partial\psi^*}{\partial t} + \psi^*\frac{\partial\psi}{\partial t} + \frac{i\hbar}{2m}\left(\psi\Delta\psi^* - \psi^*\Delta\psi\right) = 0$$

$$\psi\left(\frac{\partial\psi^*}{\partial t} + \frac{i\hbar}{2m}\Delta\psi^*\right) + \psi^*\left(\frac{\partial\psi}{\partial t} - \frac{i\hbar}{2m}\Delta\psi\right) = 0$$

$$\psi\left[-\frac{V(x)\psi^*}{i\hbar}\right] + \psi^*\left[\frac{V(x)\psi}{i\hbar}\right] = 0$$

$$\frac{1}{i\hbar}\left[\left[-\psi V(x)\psi^*\right] + \left[\psi^* V(x)\psi\right]\right] = 0$$

In the above equation V(x) is an Hermitian Operator, Y(x,t) is an eigenfunction of H and the eigenvalues are real.

[II-7] Free expansion of a gas .

$$dU = \delta Q + \delta W = 0 \qquad \begin{array}{l}\delta Q = 0 \\ \delta W = 0\end{array} \ \text{~~isothermal~~ adiabatic sudden expansion}$$

$$dU = T\,dS + p\,dV \qquad U = U(T,V) = \text{const.}$$

$$= T\left(\left(\frac{\partial S}{\partial T}\right)_V dT + \left(\frac{\partial S}{\partial V}\right)_T dV - p\,dV\right)$$

$c_V = T\left(\frac{\partial S}{\partial T}\right)_V$

$\left(\frac{\partial S}{\partial V}\right)_T = \left(\frac{\partial p}{\partial T}\right)_V$

$$= c_V\,dT + \left(T\left(\frac{\partial p}{\partial T}\right)_V - p \right) dV = 0$$

$$\rightsquigarrow \quad dT = \frac{1}{c_V}\left(p - T\left(\frac{\partial p}{\partial T}\right)_V \right) dV$$

(a) $\quad pV = nRT$

$$p = \frac{nRT}{V}$$

$$\left(\frac{\partial p}{\partial T}\right)_V = \frac{nR}{V} \quad \Rightarrow \quad dT = \frac{1}{c_V}\left(p - \frac{TnR}{V}\right) dV = 0$$

$$dT = 0 \rightsquigarrow T_F = T_i$$

(b) $\quad b\left(p + \frac{a}{V^2}\right) = nRT$

$$p = \frac{nRT}{b} - \frac{a}{V^2 b}$$

$$\left(\frac{\partial p}{\partial T}\right)_V = \frac{nR}{b} \quad \Rightarrow \quad dT = \frac{1}{c_V}\left[\frac{nRT}{b} - \frac{a}{V^2} - \frac{TnR}{b}\right] dV$$

$$dT = -\frac{1}{c_V}\frac{a}{V^2}\,dV$$

$$\int_{T_i}^{T_F} dT = -\frac{1}{c_V}\, a \int_{V_i}^{V_F} \frac{dV}{V^2} \qquad \begin{array}{l} V_i = V_0 \\ V_F = 2V_0 \end{array}$$

$$T_F - T_i = \frac{a}{c_V}\left(\frac{1}{V_F} - \frac{1}{V_i}\right) = -\frac{a}{c_V V_0}$$

II - 8

$$F(T,V,N) = F^{ideal}(T,V,N) - \frac{2}{3} N e^2 \left(\frac{4\pi N e^2}{V k_B T}\right)^{1/2}$$

a) $P = -\left(\frac{\partial F}{\partial V}\right)_{T,N} = \underbrace{-\left(\frac{\partial F^{ideal}}{\partial V}\right)_{T,N}}_{P^{ideal}(T,V)} - \frac{2}{3} N e^2 \left(+\frac{1}{2}\frac{1}{V}\right)\left(\frac{4\pi N e^2}{V k_B T}\right)^{1/2}$

$$= \frac{N k_B T}{V} - \frac{1}{3}\frac{N e^2}{V}\left(\frac{4\pi N e^2}{V k_B T}\right)^{1/2}$$

b) $F = E - TS$

$E = F + TS$ and $S = -\left(\frac{\partial F}{\partial T}\right)_{V,N}$

$$S = -\left(\frac{\partial F}{\partial T}\right)_{V,N} = -\left(\frac{\partial F^{ideal}}{\partial T}\right)_{V,N} - \frac{2}{3} N e^2 \left(+\frac{1}{2}\frac{1}{T}\right)\left(\frac{4\pi N e^2}{V k_B T}\right)^{1/2}$$

$$= S^{ideal} - \frac{1}{3}\frac{N e^2}{T}\left(\frac{4\pi N e^2}{V k_B T}\right)^{1/2}$$

$$E = F + TS = F^{ideal} - \frac{2}{3} N e^2 \left(\frac{4\pi N e^2}{V k_B T}\right)^{1/2} + T S^{ideal} - \frac{1}{3} N e^2 \left(\frac{4\pi N e^2}{V k_B T}\right)^{1/2}$$

$$= \underbrace{F^{ideal} + T S^{ideal}}_{E^{ideal}} - N e^2 \left(\frac{4\pi N e^2}{V k_B T}\right)^{1/2} =$$

$$= E^{ideal} - N e^2 \left(\frac{4\pi N e^2}{V k_B T}\right)^{1/2} = \frac{3}{2} N k_B T - N e^2 \left(\frac{4\pi N e^2}{V k_B T}\right)^{1/2}$$

c) $C_V = \left(\frac{\partial E}{\partial T}\right)_{V,N} = \frac{3}{2} N k_B + \frac{N e^2}{2T}\left(\frac{4\pi N e^2}{V k_B T}\right)^{1/2}$

other possible way to obtain C_V:

$$C_V = \left(\frac{\delta Q}{\delta T}\right)_{V,N} = T\left(\frac{\partial S}{\partial T}\right)_{V,N} = -T\left(\frac{\partial^2 F}{\partial T^2}\right)_{V,N} =$$

$$= -T\left\{\left(\frac{\partial^2 F^{ideal}}{\partial T^2}\right)_{V,N} - \frac{2}{3}\cdot\frac{1}{2}\cdot\frac{3}{2}\frac{N e^2}{T^2}\left(\frac{4\pi N e^2}{V k_B T}\right)^{1/2}\right\} =$$

$$= C_V^{ideal} + \frac{N e^2}{2T}\left(\frac{4\pi N e^2}{V k_B T}\right)^{1/2} = \frac{3}{2} N k_B + \frac{N e^2}{2T}\left(\frac{4\pi N e^2}{V k T}\right)^{1/2}$$

$\boxed{\text{II}-9}$ 3-dimensional Einstein solid

ensemble of 3N distinguishable independent quantum oscillator $\varepsilon_i = \hbar\omega_0 (n_i + \frac{1}{2})$ $n_i = 0, 1, 2, \ldots$

a) $Z_{3N} = Z_1^{3N}$

$$Z_1 = \sum_{n=0}^{\infty} e^{-\beta\hbar\omega_0 (n+\frac{1}{2})} = e^{-\beta\frac{\hbar\omega_0}{2}} \sum_{n=0}^{\infty} e^{-\beta\hbar\omega_0 n} =$$

$$= e^{-\beta\frac{\hbar\omega_0}{2}} \frac{1}{1-e^{-\beta\hbar\omega_0}} \qquad \beta = \frac{1}{k_B T}$$

$$\langle E \rangle = -\frac{\partial}{\partial\beta} \ln Z_{3N} = -3N \frac{\partial}{\partial\beta} \ln Z_1$$

$$\frac{\partial}{\partial\beta} \ln Z_1 = \frac{\partial}{\partial\beta} \left\{ -\beta\frac{\hbar\omega_0}{2} - \ln\left(1-e^{-\beta\hbar\omega_0}\right) \right\} = -\frac{\hbar\omega_0}{2} - \frac{e^{-\beta\hbar\omega_0}}{1-e^{-\beta\hbar\omega_0}} \cdot \hbar\omega_0$$

$$\langle E \rangle = \underbrace{\frac{3N}{2}\hbar\omega_0}_{\text{ground-state energy}} + \frac{3N\hbar\omega_0}{e^{\beta\hbar\omega_0}-1}$$

internal energy of Einstein solid

$$C_N = \left(\frac{\partial E}{\partial T}\right)_N = \left(\frac{\partial E}{\partial\beta}\right)_N \frac{\partial\beta}{\partial T} = -\frac{1}{k_B T^2}\left(\frac{\partial E}{\partial\beta}\right)_N = \frac{3N\hbar\omega_0}{k_B T^2} \frac{\hbar\omega_0 \, e^{\beta\hbar\omega_0}}{\left(e^{\beta\hbar\omega_0}-1\right)^2}$$

$$= 3N k_B \left(\frac{\hbar\omega_0}{k_B T}\right)^2 \frac{e^{\frac{\hbar\omega_0}{k_B T}}}{\left[e^{\frac{\hbar\omega_0}{k_B T}}-1\right]^2} =$$

$$= 3N k_B \left(\frac{\Theta_E}{T}\right)^2 \frac{e^{\Theta_E/T}}{\left(e^{\Theta_E/T}-1\right)^2} \qquad \text{where } \Theta_E = \frac{\hbar\omega_0}{k_B}$$

II-9 cont'd

b) high-temperature limit $\frac{\Theta_E}{T} \ll 1$

$$e^{\Theta_E/T} \simeq 1 + \frac{\Theta_E}{T} + \dots$$

$$C_N \cong 3Nk_B\left(\frac{\Theta_E}{T}\right)^2 \frac{1+\frac{\Theta_E}{T}+\dots}{\left(1+\frac{\Theta_E}{T}+\dots-1\right)^2} =$$

$$= 3Nk_B\left(\frac{\Theta_E}{T}\right)^2 \frac{1+\Theta_E/T+\dots}{\left(\frac{\Theta_E}{T}\right)^2+\dots} \simeq 3Nk_B + \mathcal{O}\left(\frac{\Theta_E}{T}\right)$$

$$\boxed{\simeq 3Nk_B}$$

i.e., for high-temperatures C_N approaches a constant

↳ Dulong-Petit

classical behavior:

3N classical oscillator

equipartition theorem: $2 \times \frac{1}{2}k_BT = \langle E \rangle$ per oscillator

$3Nk_BT = \langle E \rangle$ for overall

$$C_N^{\text{classical}} = \frac{\partial\langle E\rangle}{\partial T} = 3Nk_BT$$

c) low-temperature behavior $\frac{\Theta_E}{T} \gg 1$

$$C_N \simeq 3Nk_B\left(\frac{\Theta_E}{T}\right)^2 e^{-\Theta_E/T}$$

non-analytic at $T=0$

exponential vanishes much faster than power law diverges as $T \to$

Thus

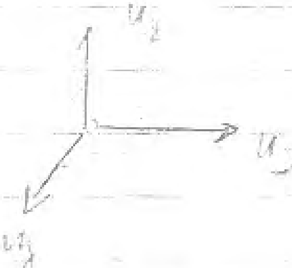
$$C_N \xrightarrow[T\to 0]{} 0 \quad \text{specific heat vanishes}$$

$\boxed{\text{II-10}}$ $\varepsilon_p = cp$ $\qquad V = L^3 \qquad (s = 1/2)$

$$\varepsilon = cp = c\hbar k = c\hbar \frac{\pi}{L}(n_x^2 + n_y^2 + n_z^2)^{1/2}$$

$$\left(\begin{array}{l} k = |\vec{k}| = (k_x^2 + k_y^2 + k_z^2)^{1/2} \\ k_x = \frac{\pi}{L} n_x \qquad n_x = 1, 2, \ldots \\ \text{etc.} \end{array} \right)$$

$$\varepsilon = \frac{ch}{2L} \underbrace{(n_x^2 + n_y^2 + n_z^2)^{1/2}}_{\equiv n} = \frac{ch}{2L} n$$



$n_x, n_y, n_z > 0$

$$N(\varepsilon) = \frac{1}{8}\frac{4\pi}{3} n^3$$

$$= \frac{1}{8}\frac{4\pi}{3}\left(\frac{2L}{ch}\right)^3 \varepsilon^3 = \frac{1}{8}\frac{4\pi}{3}\frac{8L^3}{c^3h^3}\varepsilon^3 = \frac{4\pi}{3}\frac{V}{c^3h^3}\varepsilon^3$$

$$g(\varepsilon) = \underset{(\text{spin})}{(2s+1)}\frac{\partial N}{\partial \varepsilon} = 2 \cdot \frac{4\pi}{3}\frac{V}{c^3h^3} 3\varepsilon^2 = \boxed{\frac{8\pi V}{c^3h^3}\varepsilon^2}$$

$\boxed{T = 0}$

$$N = \int_0^{\varepsilon_F} g(\varepsilon)\, d\varepsilon = \frac{8\pi V}{c^3h^3}\frac{\varepsilon_F^3}{3} = \frac{8\pi V}{3c^3h^3}\varepsilon_F^3$$

$$U_0 = \int_0^{\varepsilon_F} \varepsilon\, g(\varepsilon)\, d\varepsilon = \frac{8\pi V}{c^3h^3}\frac{\varepsilon_F^4}{4} = \frac{2\pi V}{c^3h^3}\varepsilon_F^4$$

eliminate ε_F: $\qquad \varepsilon_F = \left(\frac{3c^3h^3}{8\pi}\right)^{1/3} \left(\frac{N}{V}\right)^{1/3}$

$$\varepsilon_F = \left(\frac{3}{8\pi}\right)^{1/3} c\, h \left(\frac{N}{V}\right)^{1/3}$$

$$U_0 = \frac{2\pi V}{c^3h^3}\,\varepsilon_F^{\,4} = \frac{2\pi V}{c^3h^3}\left(\frac{3}{8\pi}\right)^{4/3} c^4h^4\left(\frac{N}{V}\right)^{4/3}$$

$$= 2\pi V\left(\frac{3}{8\pi}\right)^{4/3} c\, h \left(\frac{N}{V}\right)^{4/3} = 2\pi\, h\, c\left(\frac{3}{8\pi}\right)^{4/3} \frac{N^{4/3}}{V^{1/3}}$$

Recall, $PV = \frac{U}{3}$ for 3-dim extreme rel. gas for any temperature

Thus, $P_0 = \frac{U_0}{3V}$

$$P_0 = \frac{U_0}{3V} = \frac{2\pi h c}{3}\left(\frac{3}{8\pi}\right)^{4/3}\left(\frac{N}{V}\right)^{4/3} = \frac{hc}{4}\left(\frac{3}{8\pi}\right)^{1/3}\left(\frac{N}{V}\right)^{4/3}$$

or, Alternatively

$$P = -\left(\frac{\partial F}{\partial V}\right)_{T,N} \quad \text{where } F = U - TS$$

Thus, at $T = 0$

$$P_0 = -\left(\frac{\partial F_0}{\partial V}\right)_N = -\left(\frac{\partial U_0}{\partial V}\right)_N = \frac{2\pi}{3} hc\left(\frac{3}{8\pi}\right)^{4/3}\left(\frac{N}{V}\right)^{4/3} = \frac{hc}{4}\left(\frac{3}{8\pi}\right)^{1/3}\left(\frac{N}{V}\right)^{4/3}$$