Physics PhD Qualifying Examination
Part I – Monday, January 14, 2013

Name:_____
(please print)
Identification Number:_____

STUDENT: Designate the problem numbers that you are handing in for grading in the appropriate left hand boxes below. Initial the right hand box.
PROCTOR: Check off the right hand boxes corresponding to the problems received from each student. Initial in the right hand box.

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Student's initials

problems handed in:

Proctor's initials

INSTRUCTIONS FOR SUBMITTING ANSWER SHEETS

1. DO NOT PUT YOUR NAME ON ANY ANSWER SHEET. EXAMS WILL BE COLLATED AND GRADED BY THE ID NUMBER ABOVE.
2. Use at least one separate preprinted answer sheet for each problem. Write on only one side of each answer sheet.
3. Write your **identification number** listed above, in the appropriate box on each preprinted answer sheet.
4. Write the **problem number** in the appropriate box of each preprinted answer sheet. If you use more than one page for an answer, then number the answer sheets with both problem number and page (e.g. Problem 9 – Page 1 of 3).
5. Staple together all the pages pertaining to a given problem. Use a paper clip to group together all eight problems that you are handing in.
6. A passing distribution for the individual components will normally include at least three passed problems (from problems 1-5) for Mechanics and three problems (from problems 6-10) for Electricity and Magnetism.
7. **YOU MUST SHOW ALL YOUR WORK.**

[I-1] **[10]**

It is well known that if you drill a small tunnel through the solid non-rotating Earth of uniform density ρ from Rensselaer Polytechnic Institute through the Earth's center to its antipode location on the other side, and drop a small stone into the hole, it will be seen at the antipode location after a time $T_1 = \dfrac{\pi}{\omega_0}$, where ω_0 is a constant. (Neglect friction, air resistance, melting temperatures, etc.) The gravitational constant is G .

(a) Obtain the equation of motion for the small stone inside the Earth.

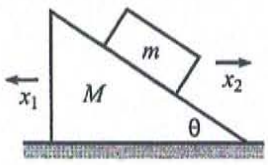
(b) Express ω_0 in terms of ρ and G .

(c) Now instead of dropping the stone, you throw it into the hole with an initial velocity v_0. How big should v_0 be, so that it now appears at the antipode location after a time $T_2 = T_1/2$? Your answer should be given in terms of ω_0 and R, the radius of the Earth

[I-2] **[10]**

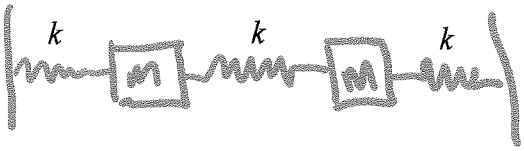
A block of mass m is held motionless on a frictionless plane of mass M and an angle of inclination θ. The plane rests on a frictionless horizontal surface. The block is released.

(a) Find the Lagrangian L of the mass and the plane and derive Lagrange's equations of motion.
(b) What is the horizontal acceleration of the plane?



[I-3] **[10]**

Two masses of mass m are connected via springs of spring constant k, as shown:



Determine the frequencies and eigenvectors associated with each of the normal modes of the system. (The masses are restricted to move horizontally only, ignore gravity.)

[I-4] **[10]**

A billiard ball (*solid* uniform sphere with radius R) at rest is hit by the cue at a height h *above the center* of the ball. Assume that during this short impulse, the force exerted by the cue on the billiard ball is much greater than all other forces.

What should h be so that the ball rolls without sliding right from the start? Your answer must be expressed in terms of R, and as always, you must show all your work to get credit. (The flat surface on which the ball rolls is horizontal, and the cue is parallel to this surface. The moment of inertia of a uniform solid sphere with mass M and radius R is $I = \frac{2}{5}MR^2$.)

[I-5] **[10]**

Consider the *relativistic* motion of a particle with (rest) mass m in one (spatial) dimension in the presence of a *time-dependent* force $F(t) = F_o e^{-t/\tau}$, where $F_o > 0$ and $\tau > 0$ are constants. The initial velocity of the particle is $v(0) = 0$. The speed of light is c.

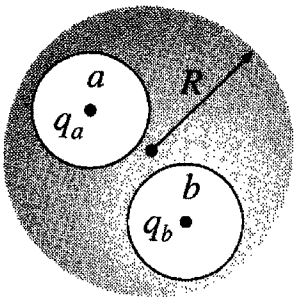
(a) Starting from the relativistic equation of motion, find $v(t)$, the velocity of the particle as a function of time.

(b) Find the limit (or terminal) velocity $\lim_{t\to\infty} v(t)$.

[I-6] **[10]**

Two spherical cavities of radii a and b are hollowed out from the interior of a (neutral) conducting sphere of radius R. At the center of each cavity a point charge is placed – call these charges q_a and q_b.

(a) Find the surface charges σ_a, σ_b and σ_R.
[σ_a is the surface charge on the surface of cavity a, σ_b is the surface charge on the surface of cavity b, σ_R and is the surface charge on the conducting sphere.]
(b) What is the electric field outside the conducting sphere?
(c) What is the electric field inside each cavity?
(d) What is the force on q_a and q_b?
(e) Which of these answers [**(a)** – **(d)**] would change if a third charge, q_c, were brought near the conductor?



[I-7] **[6,4]**

A parallel plate capacitor consists of two circular plates of area A (radius a) with vacuum between them. It is connected to a battery of constant emf $\mathcal{E}$. The plates are then *slowly* oscillated so that they remain parallel but the separation d between them is varied as

$d = d_0 + d_1 \sin(\omega t)$.

(a) Find the magnetic field H between the plates produced by the displacement current, as a function of the perpendicular radial distance r from the axis connecting the centers of the two plates.

(b) Similarly, find H if the capacitor is first disconnected from the battery and then the plates are oscillated in the same manner.

[I-8] **[6,4]**

A long straight wire of radius b carries a current I in response to a voltage V between the ends of the wire.

(a) Calculate the Poynting vector $\boldsymbol{S}$ for this DC voltage inside the wire.
(b) Obtain the energy flux per unit length at the surface of the wire. Compare this result with the Joule heating of the wire and comment on the physical significance. Is the Joule heating (IE) equal to the total incoming $\boldsymbol{S}$ flux? Is this in agreement with Poyntings theorem? State Poynting's theorem.

[I-9] **[10]**

Show that in the collision of two nonrelativistic, spinless, identical particles, the emission of electric and magnetic dipole radiation does not occur, according to classical radiation theory.

[I-10] **[10]**

An infinitely long perfectly conducting straight wire of radius r carries a constant current I and charge density zero as seen by a fixed observer A. The current is due to an electron stream of uniform density moving with high (relativistic) velocity U. A second observer B travels parallel to the wire with high (relativistic) velocity v. As seen by the observer B:
(a) What is the electromagnetic field?
(b) What is the charge density in the wire implied by this field?
(c) With what velocities do the electron and ion streams move?

Part I

I-1 Solution

Let r be the distance of the stone, of mass m, from the center of the earth. The gravitational force on it is $F = -\frac{Gm4\pi r^3\rho}{3r^2} = -\omega_0^2 mr$, where $\omega_0 = \sqrt{\frac{4\pi G\rho}{3}}$, ρ being the density of the uniform earth. The equation of the motion of the stone is then

$$\ddot{r} = -\omega_0^2 r \ .$$

Thus the stone executes simple harmonic motion with a period $T = \frac{2\pi}{\omega_0}$. Then if the stone starts from rest at Buffalo, it will reach Olaffub after a time $T_1 = \frac{T}{2} = \frac{\pi}{\omega_0}$.

The solution of the equation of motion is

$$r = A\cos(\omega t + \varphi) \ .$$

Suppose now the stone starts at $r = R$ with initial velocity $\dot{r} = -v_0$. We have

$$R = A\cos\varphi, \qquad -v_0 = -A\omega_0 \sin\varphi \ ,$$

giving

$$\varphi = \arctan\left(\frac{v_0}{R\omega_0}\right), \qquad A = \sqrt{R^2 + \left(\frac{v_0}{\omega_0}\right)^2} \ .$$

To reach Olaffub at $t = \frac{T_1}{2} = \frac{\pi}{2\omega_0}$, we require

$$-R = \sqrt{R^2 + \left(\frac{v_0}{\omega_0}\right)^2}\cos\left(\frac{\pi}{2} + \varphi\right) = -\sqrt{R^2 + \left(\frac{v_0}{\omega_0}\right)^2}\sin\varphi \ .$$

As $\sin^2\varphi + \cos^2\varphi = 1$, we have

$$\frac{R^2}{R^2 + \left(\frac{v_0}{\omega_0}\right)^2} + \frac{R^2}{R^2 + \left(\frac{v_0}{\omega}\right)^2} = 1 \ ,$$

giving

$$v_0 = R\omega_0 \ .$$

I-2 Solution:

(4) **Solutions**

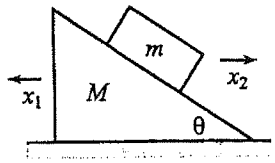


Fig. 6.37

Moving plane

Let x_1 be the horizontal coordinate of the plane (with positive x_1 to the left), and let x_2 be the horizontal coordinate of the block (with positive x_2 to the right); see Fig. 6.37. The relative horizontal distance between the plane and the block is $x_1 + x_2$, so the height fallen by the block is $(x_1 + x_2)\tan\theta$. The Lagrangian is therefore

$$L = \frac{1}{2}M\dot{x}_1^2 + \frac{1}{2}m\Big(\dot{x}_2^2 + (\dot{x}_1 + \dot{x}_2)^2\tan^2\theta\Big) + mg(x_1 + x_2)\tan\theta. \tag{6.99}$$

The equations of motion obtained from varying x_1 and x_2 are

$$\begin{aligned} M\ddot{x}_1 + m(\ddot{x}_1 + \ddot{x}_2)\tan^2\theta &= mg\tan\theta, \\ m\ddot{x}_2 + m(\ddot{x}_1 + \ddot{x}_2)\tan^2\theta &= mg\tan\theta. \end{aligned} \tag{6.100}$$

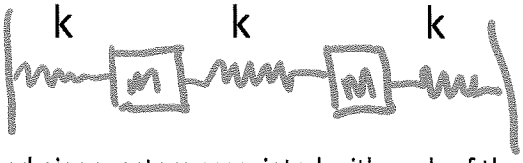
Note that the difference of these two equations immediately yields conservation of momentum, $M\dot{x}_1 - m\dot{x}_2 = 0 \Longrightarrow (d/dt)(M\dot{x}_1 - m\dot{x}_2) = 0$. Equations (6.100) are two linear equations in the two unknowns, $\ddot{x}_1$ and $\ddot{x}_2$, so we can solve for $\ddot{x}_1$. After a little simplification, we arrive at

$$\ddot{x}_1 = \frac{mg\sin\theta\cos\theta}{M + m\sin^2\theta}. \tag{6.101}$$

For some limiting cases, see the remarks in the solution to Problem 3.8.

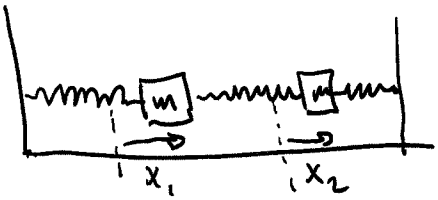
I-3 Normal Modes

Two masses of mass m are connected via springs of spring constant k, as shown:



Determine the frequencies and eigenvectors associated with each of the normal modes of the system.

Solution:



From Hooke's Law,

$$F_1 = -kx_1 + k(x_2 - x_1) = m\ddot{x}_1, \qquad F_2 = -kx_2 + k(x_1 - x_2) = m\ddot{x}_2$$

$$\ddot{x}_1 = -\omega^2 x_1, \quad \ddot{x}_2 = -\omega^2 x_2,$$

Solve for ω,

$$\begin{Vmatrix} -\frac{2k}{m} - \omega^2 & \frac{k}{m} \\ \frac{k}{m} & -\frac{2k}{m} - \omega^2 \end{Vmatrix} = 0$$

$$\left(\frac{2k}{m} - \omega^2\right)^2 = \left(\frac{k}{m}\right)^2$$

$$\frac{2k}{m} - \omega^2 = \pm\frac{k}{m}, \quad \omega = \sqrt{\frac{2k}{m} \mp \frac{k}{m}}, \qquad \boxed{\omega_- = \sqrt{\frac{k}{m}},\ \omega_+ = \sqrt{\frac{3k}{m}}}$$

eigenvectors: $\begin{pmatrix} -\frac{2k}{m} & \frac{k}{m} \\ \frac{k}{m} & -\frac{2k}{m} \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix} = -\omega^2\begin{pmatrix} A \\ B \end{pmatrix} \rightarrow \quad -\frac{2k}{m}A + \frac{k}{m}B = -\omega^2 A \quad , \quad \frac{k}{m}A - \frac{2k}{m}B = -\omega^2 B$

$\omega_- = \sqrt{\frac{k}{m}}$: $\quad -\frac{2k}{m}A + \frac{k}{m}B = -\frac{k}{m}A, \ A = B$

$\omega_+ = \sqrt{\frac{3k}{m}}$: $\quad -\frac{2k}{m}A + \frac{k}{m}B = -\frac{3k}{m}A, \ A = -B$

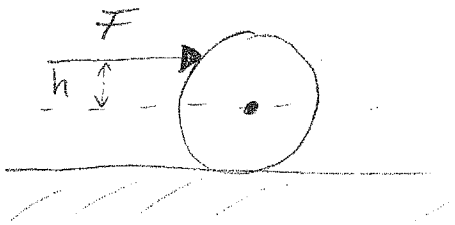
$\boxed{v_- = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}\text{ (symmetric mode)}}$

$\boxed{v_+ = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}\text{ (antisymmetric mode)}}$

I-4 [10]

A billiard ball (*solid* uniform sphere with radius R at rest) is hit by the cue at a height h *above the center* of the ball. Assume that during this short impulse, the force exerted by the cue on the billiard ball is much greater than all other forces.

What should h be so that the ball rolls without sliding right from the start? Your answer must be expressed in terms of R, and as always, you must show all your work to get credit. (The flat surface on which the ball rolls is horizontal, and the cue is parallel to this surface. The moment of inertia of a uniform solid sphere with mass M and radius R is $I = \frac{2}{5}MR^2$.)



$(F \gg F_{friction})$

Short impulse: $\begin{cases} F\,\Delta t = M v_0 \\ F h\,\Delta t = I \omega_0 \end{cases}$

$\Rightarrow \boxed{M v_0 h = I \omega_0}$

Condition for rolling without sliding from the "short": $v_0 = \omega_0 R$

$\Rightarrow M \omega_0 R h = I \omega_0$

$h = \frac{I}{MR} = \frac{\frac{2}{5}MR^2}{MR} = \boxed{\frac{2}{5}R}$

I-5 [10]

Consider the *relativistic* motion of a particle with (rest) mass m in one (spatial) dimension in the presence of a *time-dependent* force $F(t) = F_o e^{-t/\tau}$, where F_o and τ are constants. The initial velocity of the particle is $v(0) = 0$. The speed of light is c. $(F_o > 0, \tau > 0)$

(a) Starting from the relativistic equation of motion, find $v(t)$, the velocity of the particle as a function of time.

(b) Find the limit (or terminal) velocity $\lim_{t\to\infty} v(t)$.

a)

$$F(t) = \frac{d}{dt}\left(\frac{mv}{\sqrt{1-v^2/c^2}}\right) \qquad v(0) = 0$$

$$\int_0^t F(t')dt' = \frac{mv(t)}{\sqrt{1-\frac{v^2(t)}{c^2}}}$$

$$\int_0^t F_o e^{-t'/\tau} dt' = \frac{mv(t)}{\sqrt{1-\frac{v^2(t)}{c^2}}}$$

$$F_o\tau\left(1-e^{-t/\tau}\right) = \frac{mv(t)}{\sqrt{1-\frac{v^2(t)}{c^2}}}$$

$$\Rightarrow \quad v(t) = \frac{F_o\tau\left(1-e^{-t/\tau}\right)}{\sqrt{m^2 + \frac{F_o^2\tau^2\left(1-e^{-t/\tau}\right)^2}{c^2}}} = \boxed{\frac{\frac{F_o\tau}{m}\left(1-e^{-t/\tau}\right)}{\sqrt{1+\frac{F_o^2\tau^2}{m^2c^2}\left(1-e^{-t/\tau}\right)^2}}}$$

b)

$$\lim_{t\to\infty} v(t) = \boxed{\frac{\frac{F_o\tau}{m}}{\sqrt{1+\frac{F_o^2\tau^2}{m^2c^2}}}}$$

I-6 Solution:

Electricity and Magnetism
I-6 Electrostatics or Boundary Value

(a) $\boxed{\sigma_a = -\frac{q_a}{4\pi a^2};}$ $\boxed{\sigma_b = -\frac{q_b}{4\pi b^2};}$ $\boxed{\sigma_R = \frac{q_a + q_b}{4\pi R^2}.}$

(b) $\boxed{\mathbf{E}_{\text{out}} = \frac{1}{4\pi\epsilon_0}\frac{q_a + q_b}{r^2}\hat{\mathbf{r}},}$ where $\mathbf{r}$ = vector from center of large sphere.

(c) $\boxed{\mathbf{E}_a = \frac{1}{4\pi\epsilon_0}\frac{q_a}{r_a^2}\hat{\mathbf{r}}_a, \quad \mathbf{E}_b = \frac{1}{4\pi\epsilon_0}\frac{q_b}{r_b^2}\hat{\mathbf{r}}_b,}$ where $\mathbf{r}_a$ ($\mathbf{r}_b$) is the vector from center of cavity a (b).

(d) $\boxed{\text{Zero.}}$

(e) σ_R changes (but not σ_a or σ_b); $\mathbf{E}_{\text{outside}}$ changes (but not $\mathbf{E}_a$ or $\mathbf{E}_b$); force on q_a and q_b still zero.

Problem I-7

(a) $\mathcal{E} = Ed$, $E = \mathcal{E}/d$.

$$J_d = \epsilon_0 \frac{dE}{dt} = -\epsilon_0 \frac{\mathcal{E}}{d^2} \frac{dd}{dt}$$

$$\frac{dd}{dt} = d_1 \omega \cos \omega t, \quad d = d_0 + d_1 \sin \omega t$$

Ampère's Law gives

$$\vec{H} = \hat{\varphi} \frac{A}{2\pi r} J_d \qquad r > a$$

$$\vec{H} = \hat{\varphi} \frac{A}{2\pi r} \frac{r^2}{a^2} J_d \qquad r < a$$

(b) $Q = \text{const.} \quad E = Q/\epsilon_0 = \text{const.}$

$\therefore J_d = 0 \Rightarrow \vec{H} = 0.$

I-8. Solution

a) Let us calculate the flux of the Poynting vector. Introduce cylindrical coordinates with unit vectors $\mathbf{e}_\rho$, $\mathbf{e}_\theta$, and $\hat{\mathbf{z}}$. Current flows along the wire in the z direction and the electric field $\mathbf{E} = E\hat{\mathbf{z}}$. Using one of Maxwell's equations in vacuum, the fact that conditions are stationary, and Stokes' theorem,

$$\nabla \times \mathbf{B} = \frac{4\pi}{c}\mathbf{J} + \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t}$$

$$\int_A \nabla \times \mathbf{B} \cdot d\mathbf{A} = \oint_C \mathbf{B} \cdot d\mathbf{l} = \frac{4\pi}{c}\int_A \mathbf{J} \cdot d\mathbf{A}$$

where J is the current density and A is the surface. At any given radius

r, B_θ is constant, so we have

$$2\pi r B = \frac{4\pi}{c} J\pi r^2 \qquad \mathbf{B} = \frac{2\pi}{c} Jr\mathbf{e}_\theta$$

$$\mathbf{S} = \frac{c}{4\pi}\mathbf{E} \times \mathbf{B} = \frac{c}{4\pi}\frac{2\pi}{c} JrE\,(\hat{\mathbf{z}} \times \mathbf{e}_\theta) = -\frac{1}{2} JrE\mathbf{e}_\rho$$

Using the relation between current density and total current $J = I/(\pi b^2)$:

$$\mathbf{S} = -\frac{IEr}{2\pi b^2}\mathbf{e}_\rho \qquad \mathbf{S}(b) = -\frac{IE}{2\pi b}\mathbf{e}_\rho$$

b) The Poynting flux per unit length is then $\mathbf{S} \cdot 2\pi\mathbf{b} = -IE$. So the flux enters the wire, and we see that the dissipated power per unit length IE is equal to the total incoming S-flux, in agreement with Poynting's theorem:

$$\frac{\partial u}{\partial t} = -\mathbf{J} \cdot \mathbf{E} - \nabla \cdot \mathbf{S}$$

where u is the energy density. Under stationary conditions such as ours

$$\frac{\partial u}{\partial t} = 0$$

and we have

$$\int_V \mathbf{J} \cdot \mathbf{E}\, d^3x = -\int_V \nabla \cdot \mathbf{S}\, dV = -\int_A \mathbf{S} \cdot d\mathbf{A} = IE$$

I-9. Solution

Electric-dipole radiation is proportional to $|d\mathbf{D}/dt|^2$, where $\mathbf{D}$ is the electric dipole moment. But $\mathbf{D} = e(\mathbf{r}_1 + \mathbf{r}_2)$, and this has zero time derivative because it is proportional to the center-of-mass vector. The magnetic moment is

$$\mathbf{M} = \frac{1}{2c}\sum_{i=1}^{2}(\mathbf{r}_i \times \mathbf{J}_i) = \frac{e}{2c}\sum_{i=1}^{2}(\mathbf{r}_i \times \mathbf{v}_i) = \frac{e}{2mc}\mathbf{L}.$$

Thus $\mathbf{M}$ is proportional to the angular momentum of the system, which is constant. Magnetic-dipole radiation is proportional to $|d\mathbf{M}/dt|^2$, and so is equal to zero.

I-10 *solution:*

(a) Let Σ and Σ' be the rest frames of observers A and B respectively, the common x-axis being along the axis of the conducting wire, which is fixed in Σ.

In Σ : $\rho = 0,\ j = \frac{i}{\pi r_0^2}\hat{x},$

yielding fields, $E = 0, B(r) = \frac{\mu_0 I}{2\pi r}\hat{\phi}.$

To find fields in Σ', perform Lorentz transformation.

$E'_{||} = E_{||} = 0,\quad B'_{||} = B_{||} = 0,$

$$E' = E'_{\perp} = \gamma\left(E_{\perp} + v \times B_{\perp}\right) = -\gamma v B\hat{r} = -\frac{\gamma v \mu_0 I}{2\pi r}\,\hat{r}$$

$$B' = B'_{\perp} = \gamma\left(B_{\perp} - v \times \frac{E_{\perp}}{c^2}\right) = \gamma B\hat{\phi} = \frac{\gamma \mu_0 I}{2\pi r}\hat{\phi},$$

with $\gamma = 1/\sqrt{1 - v^2/c^2}$.

(b) Let the linear charge density of the wire in Σ' be ρ', then the electric field produced by ρ' is given by Gauss' law:

$$2\pi r E' = \frac{\rho'}{\varepsilon_0},$$

$$\rho' = 2\pi r \varepsilon_0\left(-\frac{\gamma v \mu_0 I}{2\pi r}\right) = -\frac{\gamma v I}{c^2},$$ where we have used $\mu_0\varepsilon_0 = 1/c^2$

(c) In Σ the velocity of the electron stream is $v_e = -U\hat{x}$, while the ions are stationary, i.e., $v_i = 0$. Using the Lorentz transformation of velocity we have in Σ'

$$v'_e = -\frac{v + U}{1 + \frac{vU}{c^2}}\hat{x},\quad v'_i = -v\hat{x}$$

Physics PhD Qualifying Examination
Part II – Tuesday, January 15, 2013

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7. **YOU MUST SHOW ALL YOUR WORK.**

[II-1] **[10]**

Using the annihilation operator $a = \sqrt{\frac{m\omega}{2\hbar}}(\hat{x} + \frac{i}{m\omega}\hat{p})$, and creation operator $a^{\dagger} = \sqrt{\frac{m\omega}{2\hbar}}(\hat{x} - \frac{i}{m\omega}\hat{p})$, determine the normalized wavefunctions and energies of the ground state and first excited state of the simple harmonic oscillator, with potential $V(x) = \frac{1}{2}m\omega^2x^2$.

[II-2] **[10]**

Consider a particle of mass m in a one-dimensional box with infinite high walls at $x=0$ and $x=L$.

(a) Find the eigenenergies E_n and normalized eigenfunctions ϕ_n for the particle in the box.

(b) Calculate the first order correction to $E_2^{(0)}$ for the particle due to the following perturbation $H'=10^{-3}\,E_1\frac{x^2}{L^2}$. Here, $E_1 = \frac{\hbar^2\pi^2}{2mL^2}$ is a constant.

[II-3] **[10]**

A particle in a central field potential has an orbital angular momentum $l = 2\hbar$ and spin $s = \hbar$. Find the energy levels and degeneracies associated with a spin-orbit interaction term of the form $H_{SO} = A\boldsymbol{L}\cdot\boldsymbol{S}$, where A is a constant.

[II-4] **[10]**

A particle of mass m is scattered by the Yukawa potential:

$$V(r) = V_0 \frac{\exp(-\mu r)}{\mu r}, \qquad \text{with } \mu\text{=const.}$$

(a) Calculate the scattering amplitude in the first Born approximation
(b) Calculate the differential scattering cross section.
(c) Calculate the total scattering cross section.

[II-5] **[10]**

Compute the cross section for hard sphere scattering in terms of a partial wave expansion, as a function of the wave number k. The potential is infinite for $r \leq a$ and zero for $r > a$.
Hint: the wavefunction has a partial wave expansion

$$\psi(r,\theta) = A \sum_{l=0}^{\infty} \left[i^l (2l+1) j_l(kr) + \sqrt{\frac{2l+1}{4\pi}} C_l h_l^{(1)}(kr) \right] P_l(\cos\theta)$$

and your final answer will be in terms of the spherical Bessel function $j_l(kr)$ and spherical Hankel function $h_l^{(1)}(kr)$ evaluated at $r = a$. Also note that the outgoing scattered spherical wave is conveyed by the asymptotic behavior $h_l^{(1)}(kr) \sim (-i)^{l+1} e^{ikr}/kr$ at large r.

[II-6] **[10]**

A two-level system has eigenstates a and b with energies E_a and E_b, where we assume $E_b > E_a$. The system is perturbed by a time dependent potential H' with matrix elements

$$H'_{ab}(t) = H'_{ba}(t) = \lambda e^{-\alpha t^2},$$

with λ a small parameter. (Diagonal matrix elements of H' vanish.)

If the system starts in state a in the infinite past, at first order in time dependent perturbation theory, what is the probability that it will be in state b in the infinite future?

[II-7] **[10]**

Monatomic ideal gas consisting of N atoms is confined to a container of fixed volume. At the end of a process, during which heat is slowly transferred to the gas, we find that the pressure of the gas is tripled.

What is the change of the entropy of the gas? You must express your answer in terms of N and k.

[II-8] **[10]**

The Helmholtz free energy of a gas is given by $F(T,V) = -\frac{a}{3}T^4V$, where a is a positive constant. Determine the relationship between P (pressure) and V (volume) for the ***quasi-static (reversible) adiabatic*** process.

[II-9] **[2,3,2,3]**

In a static approximation, the energy of a particle depends only on the position, $E = E(x)$. Here we will take the particle to be confined to one dimension, and constrained to the positive real axis, $0 \leq x < \infty$. We will also assume that the energy is given by $E = \lambda x^m$. The classical partition function in this case is given by $Z = \int dx\, e^{-\frac{E(x)}{kT}}$. An integral that you will need to know is $\int_0^\infty dx\, e^{-(rx)^m} = \frac{1}{mr}\Gamma(1/m)$, where $\Gamma(z)$ is Euler's gamma function. For a single particle,

(a) Compute the partition function.
(b) Find the Helmholtz free energy.
(c) Find the entropy.

Now suppose we include momentum in the energy, $E = \frac{p^2}{2\mu} + \lambda x^m$, where μ is the mass of the particle.

(d) Compute the entropy in this case.

Hint: The partition function becomes $Z = \int \frac{dx\, dp}{2\pi\hbar}\, e^{-\frac{E(p,x)}{kT}}$ when momentum is included.

[II-10] **[10]**

If a magnetic field H, is applied to a gas of uncharged particles having spin $\frac{1}{2}$ and the magnetic moment μ, and obeying Fermi-Dirac statistics, the lining up of the spins produces a magnetic moment/volume. Set up the general expressions for the magnetic moment/volume at arbitrary temperature T and field H.

Then for low enough temperature, determine the magnetic susceptibility of the gas in the limit of zero magnetic field, correct to terms of order T^2.

Note: You may want to use the following integral,

$$\int_0^\infty \frac{\sqrt{E}dE}{\exp[(E-\xi)/kT]+1} = \frac{2}{3}\xi^{3/2}[1+\frac{\pi^2}{8}(\frac{kT}{\xi})^2+\ldots\,].$$

Part II

II-1 solution:

The annihilation operator acting on the ground state returns zero,

$$a|0\rangle = \sqrt{\frac{m\omega}{2\hbar}}\left(x + \frac{i}{m\omega}\frac{\hbar}{i}\frac{\partial}{\partial x}\right)\Psi_0(x) = 0,$$

$$x\Psi_0 = -\frac{\hbar}{m\omega}\frac{\partial}{\partial x}\Psi_0$$

$$\int -\frac{m\omega}{\hbar}x dx = \int \frac{1}{\Psi} d\Psi\,, \quad -\frac{\frac{1}{2}m\omega}{\hbar}x^2 = \ln(\Psi) + C$$

$$\Psi_0 = Ae^{-m\omega x^2/2\hbar}$$

The first excited state is then found from, $|1\rangle = a^\dagger|0\rangle$.

II-2 solutions

II-2

(a) eigenenergies $E_n = n^2 E_1$ with $E_1 = \frac{\hbar^2 \pi^2}{2mL^2}$

eigenfunctions $\phi_n = \sqrt{\frac{2}{L}} \sin \frac{n\pi x}{L}$

(b) first-order correction to $E_2^{(0)}$

$$E_2^{(1)} = \langle \phi_2 | H' | \phi_2 \rangle$$

$$= 10^{-3} \frac{E_1}{L^2} \cdot \frac{2}{L} \int_0^L x^2 \sin^2 \frac{2\pi x}{L} \, dx$$

$y = \frac{2\pi}{L} x$, $dy = \frac{2\pi}{L} dx$ $\quad x^2 dx = \frac{L^3}{8\pi^3} y \, dy$

$\frac{Ly}{2\pi} = x$ $\quad \frac{L dy}{2\pi} = dx$ $\quad x=0 \Rightarrow y=0$ $\quad x=L$ $y=2\pi$

$$E_2^{(1)} = 10^{-3} \frac{2E_1}{L^3} \frac{L^3}{8\pi^3} \int_0^{2\pi} y^2 \sin^2 y \, dy$$

$$E_2^{(1)} = 10^{-3} \frac{E_1}{4\pi^3} \left[2y \sin y - (y^2 - 2) \cos y \right]_0^{2\pi}$$

$$E_2^{(1)} = 10^{-3} \frac{E_1}{4\pi^3} \left[2 \cdot 2\pi \underbrace{\sin 2\pi}_{=0} - (4\pi^2 - 2) \underbrace{\cos 2\pi}_{=1} - \right.$$

$$\left. \left(- (-2 \underbrace{\cos 0}_{=1}) \right) \right]$$

$$= 10^{-3} \frac{E_1}{4\pi^3} \left[4 - (4\pi^2 - 2) - 2 \right]$$

$$= 10^{-3} \frac{E_1}{4\pi^3} (-4\pi^2) = -10^{-3} \frac{E_1}{\pi}$$

II-3 *solution:*

Choose $\{H, J^2, J_z, L^2, S^2\}$ as a complete set of mechanical variables. The wave function associated with angle and spin is ϕ_{jm_jls}, for which:

$J^2\phi_{jm_jls} = \hbar^2 j(j+1)\phi_{jm_jls}$ $\quad L^2\phi_{jm_jls} = \hbar^2 l(l+1)\phi_{jm_jls}$,

$S^2\phi_{jm_jls} = \hbar^2 s(s+1)\phi_{jm_jls}$, and $\quad J_z\phi_{jm_jls} = \hbar m_j\phi_{jm_jls}$.

$$H_{SO} = a\, L \cdot S = \frac{1}{2}A(J^2 - L^2 - S^2)$$

$E_{SO} = \frac{\hbar^2}{2}A[j(j+1) - l(l+1) - s(s+1)]$, s=1, l=2

$$E_{SO} = \frac{\hbar^2}{2}A[j(j+1) - 6 - 2]$$

$$E_{SO} = \begin{cases} 2A\hbar^2, & j = 3 \\ -A\hbar^2, & j = 2 \\ -3A\hbar^2, & j = 1 \end{cases}, \qquad degeneracy = 2(j+1) = \begin{cases} 7, & j = 3 \\ 5, & j = 2 \\ 3, & j = 1 \end{cases}$$

II-4

(a) $$f^{(1)}(\theta) = -\frac{2m}{\hbar^2}\int V(r')\,\frac{\sin Kr'}{Kr'}\,r'^2\,dr'$$

$$= -\frac{2m}{\hbar^2}\int_0^\infty \frac{V_0}{\mu}\,\frac{e^{-\mu r'}}{r'}\,\frac{\sin Kr'}{Kr'}\,r'^2\,dr'$$

$$= -\frac{2m}{\hbar^2}\frac{V_0}{\mu K}\int_0^\infty e^{-\mu r'}\sin Kr'\,dr'$$

$$= -\frac{2m}{\hbar^2}\frac{V_0}{\mu K}\left[\frac{e^{-\mu r'}}{\mu^2+K^2}\left(-\mu\sin Kr' - K\cos Kr'\right)\right]_0^\infty$$

$$= -\frac{2m}{\hbar^2}\frac{V_0}{\mu K}\left(-\frac{1}{\mu^2+K^2}(-K)\right)$$

$$f^{(1)}(\theta) = -\frac{2m}{\hbar^2}\frac{V_0}{\mu K}\frac{K}{\mu^2+K^2} = -\frac{2m}{\hbar^2}\frac{V_0}{\mu}\frac{1}{\mu^2+K^2}$$

(b) $$\frac{d\sigma}{d\theta} = |f^{(1)}(\theta)|^2 = \frac{4m^2}{\hbar^4}\frac{V_0^2}{\mu^2}\frac{1}{(\mu^2+K^2)^2} \quad \text{with } K = 2k\sin\frac{\theta}{2}$$

$$= \frac{4m^2V_0^2}{\hbar^4\mu^2}\frac{1}{(\mu^2+4k^2\sin^2\frac{\theta}{2})^2}$$

$$\int e^{+ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)$$

(c)

$$\sigma_{tot} = \int \frac{d\sigma}{d\Omega}\, d\Omega$$

$$= \frac{4m^2V_0^2}{\hbar^4\mu^2} \int_0^{2\pi}\int_0^{\pi} \frac{1}{(\mu^2 + 4k^2 \sin^2\frac{\theta}{2})^2} \sin\theta\, d\theta\, d\varphi$$

$$= \frac{8\pi m^2V_0^2}{\hbar^4\mu^2} \int_0^{\pi} \frac{\sin\theta\, d\theta}{(\mu^2 + 4k^2 \sin^2\frac{\theta}{2})^2}$$

$$= \frac{8\pi m^2V_0^2}{\hbar^4\mu^2} \int_0^{\pi} \frac{\sin\frac{\theta}{2}\cos\frac{\theta}{2}\, d\theta}{(\mu^2 + 4k^2 \sin^2\frac{\theta}{2})^2}$$

$X = 2k \sin\frac{\theta}{2}$, $\sin\frac{\theta}{2} = \frac{X}{2k}$, $\theta = 0 \Rightarrow X = 0$

$dX = 2k \cdot \frac{1}{2} \cos\frac{\theta}{2}\, d\theta$, $\cos\frac{\theta}{2}\, d\theta = \frac{dX}{k}$, $\theta = \pi \Rightarrow X = 2k$

$$\sigma_{tot} = \frac{8\pi^2 m^2V_0^2}{\hbar^4\mu^2 2k^2} \int_0^{2k} \frac{X\, dX}{(1+X^2)^2}$$

$u = 1 + X^2$

$du = 2X\, dX$

$X\, dX = \frac{du}{2}$

$X = 0 \quad u = 1$

$X = 2k \quad u = 1 + 4k^2$

$$= \frac{8\pi^2 m^2V_0^2}{\hbar^4\mu^2 2k^2} \cdot \frac{1}{2} \int_1^{1+4k^2} \frac{du}{u^2}$$

$$= \frac{4\pi^2 m^2V_0^2}{\hbar^4\mu^2 k^2} \left[-\frac{1}{u}\right]_1^{1+4k^2}$$

$$\sigma_{tot} = \frac{4\pi^2 m^2V_0^2}{\hbar^4\mu^2 k^2} \left[-\frac{1}{1+4k^2} - \left(-\frac{1}{1}\right)\right]$$

$$= \frac{4\pi^2 m^2V_0^2}{\hbar^4\mu^2 k^2} \left[1 - \frac{1}{1+4k^2}\right] = \frac{4\pi^2 m^2V_0^2}{\hbar^4\mu^2 k^2} \left[\frac{1+4k^2-1}{1+4k^2}\right]$$

$$\sigma_{tot} = \frac{4\pi^2 m^2V_0^2}{\hbar^4\mu^2 k^2} \left[\frac{4k^2}{1+4k^2}\right]$$

Problem II-5

We have the boundary condition $\Psi(a,\theta) = 0$ b/c of the infinite potential. Thus

$$i^\ell (2\ell+1)\, j_\ell(ka) + \sqrt{\frac{2\ell+1}{4\pi}}\, c_\ell\, h_\ell^{(1)}(ka) = 0$$

$$\Rightarrow c_\ell = -\frac{i^\ell (2\ell+1)\, j_\ell(ka)}{\sqrt{\frac{2\ell+1}{4\pi}}\, h_\ell^{(1)}(ka)}$$

The asymptotic behavior is

$$\Psi(r,\theta) = A\left[e^{ikz} + \sum_\ell \sqrt{(2\ell+1)}\, c_\ell \frac{e^{ikr}}{kr} P_\ell(\cos\theta)\right]$$

which is to be matched to

$$\Psi(r,\theta) = A\left[e^{ikz} + f(\theta)\frac{e^{ikr}}{r}\right]$$

$$\Rightarrow \frac{d\sigma}{d\Omega} = |f(\theta)|^2 = \sum_{\ell,\ell'} \frac{\sqrt{(2\ell+1)(2\ell'+1)}}{4\pi} \cdot c_\ell c_{\ell'}^* P_\ell(\cos\theta) P_{\ell'}(\cos\theta)\, \frac{1}{k^2}$$

$$\sigma = \frac{2\pi}{k^2} \sum_{ll'} \frac{\sqrt{(2l+1)(2l'+1)}}{4\pi} c_l c_{l'}^*$$

$$\int_{-1}^{1} (d\cos\theta)\, P_l(\cos\theta)\, P_{l'}(\cos\theta)$$

$$= \frac{2\pi}{k^2} \sum_{ll'} \frac{\sqrt{(2l+1)(2l'+1)}}{4\pi} c_l c_{l'}^* \frac{2}{2l+1} \delta_{ll'}$$

$$= \frac{1}{k^2} \sum_{l} |c_l|^2$$

with the c_l above.

Problem II - 6

$$\Psi(t) = c_a(t)\Psi_a e^{-iE_a t/\hbar} + c_b(t)\Psi_b e^{-iE_b t/\hbar}$$

$$c_b^{(1)} = -\frac{i}{\hbar}\int_{-\infty}^{t} dt' \; \lambda e^{-\alpha t'^2 + i\omega_0 t'}$$

$$c_b^{(1)}(\infty) = -\frac{i}{\hbar}\lambda\sqrt{\frac{\pi}{\alpha}}\, e^{-\omega_0^2/4\alpha}$$

$$P_{a\to b} = |c_b^{(1)}(\infty)|^2$$

$$= \frac{\lambda^2 \pi}{\alpha \hbar^2} e^{-\omega_0^2/2\alpha}$$

$$\omega_0 = (E_b - E_a)/\hbar$$

II-7 [10]

Monatomic ideal gas consisting of N atoms is confined to a container of fixed volume. At the end of a process, during which heat is slowly transferred to the gas, we find that the pressure of the gas is tripled.

What is the change of the entropy of the gas? You must express your answer in terms of N and k.

$PV = NkT$

$V = \text{const.} \Rightarrow \frac{P_2}{P_1} = \frac{T_2}{T_1} \Rightarrow \boxed{\frac{T_2}{T_1} = 3}$

$dU = T\,dS - P\,dV$ (fundamental equation)

$dU = T\,dS$ (since $P\,dV = 0$)

$dU = \frac{f}{2} Nk\,dT = \frac{3}{2} Nk\,dT$

$dS = \frac{dU}{T} = \frac{3}{2} Nk \frac{dT}{T}$

$\Delta S = \int_i^f dS = \int_{T_1}^{T_2} \frac{3}{2} Nk \frac{dT}{T} = \frac{3}{2} Nk \int_{T_1}^{T_2} \frac{dT}{T} =$

$= \frac{3}{2} Nk \ln\left(\frac{T_2}{T_1}\right) = \boxed{\frac{3}{2} Nk \ln(3)}$

Alternatively, may recall: $\Delta S = C_V \ln\left(\frac{T_2}{T_1}\right) + Nk \ln\left(\frac{V_2}{V_1}\right)$ for ideal gas

$\Rightarrow \Delta S = \frac{3}{2} Nk \ln(3)$

II-8 [10]

The Helmholtz free energy of a gas is given by $F(T,V) = -\frac{a}{3}T^4V$, where a is a positive constant. Determine the relationship between P (pressure) and V (volume) for the ***quasi-static (reversible) adiabatic*** process.

$$P = -\left(\frac{\partial F}{\partial V}\right)_T = \frac{a}{3}T^4 \quad \Rightarrow \quad T = \left(\frac{3P}{a}\right)^{1/4}$$

$$S = -\left(\frac{\partial F}{\partial T}\right)_V = \boxed{\frac{4}{3}aT^3V}$$

adiabatic reversible process. $dQ = 0$

$dQ = T\,dS \Rightarrow dS = \frac{dQ}{T} = 0 \Rightarrow$ **S = const. (isentropic process)**

$$\boxed{\frac{4}{3}aT^3V = \text{const.}} \qquad (T^3V = \text{const.})$$

$$\frac{4}{3}a\left(\frac{3P}{a}\right)^{3/4} \cdot V = \text{const.} \Rightarrow \boxed{P^{3/4} \cdot V = \text{const.}}$$

or equivalently: $P \cdot V^{4/3} = \text{const.}$

alternative solution: $U = F + TS = -\frac{a}{3}T^4V + \frac{4}{3}aT^4V = aT^4V$

first law when $dQ = 0$: $dU = -P\,dV$

$$4aT^3V\,dT + aT^4\,dV = -\frac{a}{3}T^4\,dV$$

$$4aT^3V\,dT = -\frac{4}{3}aT^4\,dV$$

$$3\frac{dT}{T} = -\frac{dV}{V}$$

$3\ln(T) = -\ln V + \text{const.} \Rightarrow T^3V = \text{const.}$, and follow solution from above for P-V curve

Problem II-9

(a) $r^m = \frac{\lambda}{kT} \Rightarrow r = \left(\frac{\lambda}{kT}\right)^{1/m}$

$$Z = \frac{1}{m\left(\frac{\lambda}{kT}\right)^{1/m}\Gamma(1/m)}$$

(b) $F = -kT \ln Z$

$$= -kT\left[\ln \frac{\Gamma(1/m)}{m(\lambda/k)^{1/m}} + \frac{1}{m}\ln T\right]$$

(c) $S = -\left(\frac{\partial F}{\partial T}\right)_V$

$$= k\left[\ln\left(\frac{\Gamma(1/m)}{m}\left(\frac{k}{\lambda}\right)^{1/m}\right) + \frac{1}{m}\ln T\right]$$

$$- (-k\not{T})\frac{1}{m}\frac{1}{\not{T}}$$

$$= k\left[\frac{1}{m} + \ln\left(\frac{\Gamma(1/m)}{m}\left(\frac{k}{\lambda}\right)^{1/m}\right) + \frac{1}{m}\ln T\right]$$

(d) $$\int \frac{dp}{2\pi\hbar} e^{-p^2/2\mu} = \sqrt{\frac{\mu}{2\pi\hbar^2}}$$

$$F \to F - \frac{1}{2} k T \ln\left(\frac{M}{2\pi k^2}\right)$$

$$S \to S + \frac{1}{2} k \ln\left(\frac{M}{2\pi k^2}\right)$$

Just adds another constant to the entropy.

II-10. Solution The energy of a particle whose magnetic moment is parallel (anti-parallel) to H, is given by

$$U_{\mp} = \frac{p^2}{2m} \mp \mu H.$$

Since the energy levels of the system are populated according to the distribution function

$$f(U) = \frac{1}{\exp{(U-\xi)/kT} + 1},$$

and the density of levels is given by $(4\pi V/h^3)\, p^2 dp$, the total number of particles N is given by

$$N = \frac{4\pi V}{h^3} \int dp\, p^2 [f(U_-) + f(U_+)] \tag{1}$$

and the magnetization/volume is

$$\frac{M}{V} = \frac{4\pi\mu}{h^3} \int dp\, p^2 [f(U_-) - f(U_+)]. \tag{2}$$

Equation (1) may be solved for ξ in terms of N, T, and H, and ξ may then be substituted in Eq. (2) to determine M/V as a function of N, T, and H.

Upon defining a new variable of integration $E = p^2/2m$ and using the low-temperature expansion formula given, we find that Eq. (2) becomes

$$\frac{M}{V} = \frac{8\pi\mu(2m^3)^{1/2}}{3h^3}\left\{(\xi+\mu H)^{3/2}\left[1 + \frac{\pi^2}{8}\left(\frac{kT}{\xi+\mu H}\right)^2\right] - (\xi-\mu H)^{3/2}\left[1 + \frac{\pi^2}{8}\left(\frac{kT}{\xi-\mu H}\right)^2\right]\right\},$$

which, after expanding in powers of H and keeping only the leading term, becomes

$$\left(\frac{M}{V}\right) = \frac{8\pi\mu^2(2m^3)^{1/2}}{h^3}\xi^{1/2} H\left\{1 - \frac{\pi^2}{24}\left(\frac{kT}{\xi}\right)^2 + \cdots\right\} + \text{terms of order } H^3.$$

Equation (1) (for $H = 0$) becomes

$$n = \frac{N}{V} = \frac{16\pi}{3h^3}(2m^3)^{1/2}\xi^{3/2}\left\{1 + \frac{\pi^2}{8}\left(\frac{kT}{\xi}\right)^2 + \cdots\right\}.$$

Solving for ξ, one obtains

$$\xi = \xi_0\left\{1 - \frac{\pi^2}{12}\left(\frac{kT}{\xi_0}\right)^2 + \cdots\right\},$$

where ξ_0 is the Fermi energy at $T = 0°\text{K}$, and $\xi_0^{3/2} = 3h^3 n/16\pi(2m^3)^{1/2}$. The susceptibility then becomes

$$\chi \equiv \frac{M}{VH} = \left(\frac{3\mu^2 n}{2\xi_0}\right)\left\{1 - \frac{\pi^2}{12}\left(\frac{kT}{\xi_0}\right)^2 + \cdots\right\}.$$