Physics PhD Qualifying Examination
Part I – Wednesday, August 26, 2015

Name:____

(please print)

Identification Number:_________

<u>STUDENT</u>: Designate the problem numbers that you are handing in for grading in the appropriate left hand boxes below. Initial the right hand box.
<u>PROCTOR</u>: Check off the right hand boxes corresponding to the problems received from each student. Initial in the right hand box.

	1	
	2	
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	8	
	9	
	10	

Student's initials

\# problems handed in:

Proctor's initials

<u>INSTRUCTIONS FOR SUBMITTING ANSWER SHEETS</u>

1. DO NOT PUT YOUR NAME ON ANY ANSWER SHEET. EXAMS WILL BE COLLATED AND GRADED BY THE ID NUMBER ABOVE.
2. Use at least one separate preprinted answer sheet for each problem. Write on only one side of each answer sheet.
3. Write your **<u>identification number</u>** listed above, in the appropriate box on each preprinted answer sheet.
4. Write the **<u>problem number</u>** in the appropriate box of each preprinted answer sheet. If you use more than one page for an answer, then number the answer sheets with both problem number and page (e.g. Problem 9 – Page 1 of 3).
5. Staple together all the pages pertaining to a given problem. Use a paper clip to group together all problems that you are handing in.
6. **There is <u>no limit on the number of problems</u> you can turn in.**
7. A passing distribution for the individual components will normally include at least three passed problems (from problems 1-5) for Mechanics and three problems (from problems 6-10) for Electricity and Magnetism.
8. **YOU MUST SHOW ALL YOUR WORK.**

I-1 [10]

A particle moves in a medium under the influence of a force,

$$F = -mk\left(v^4 + 2av^2 + a^2\right),$$

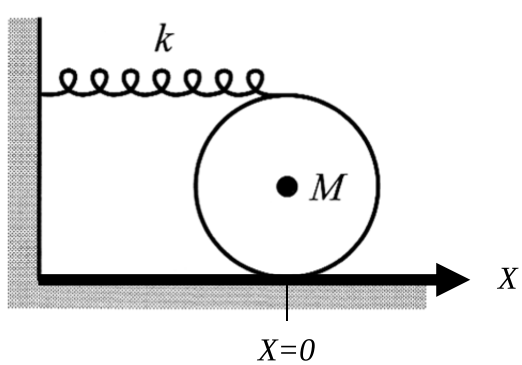
where k are a constants. There are no other forces present. The particle is initially at the origin (x=0) and is given an initial velocity of v_0. Treating the problem classically (not relativistically),

(a) What is the distance the particle travels before coming to a stop (i.e. v=0)? (Note that your answer should be given in terms of v_0, a, and k.)
(b) What is the *maximum* possible distance it can ever travel before coming to a stop?

I-2 [10]

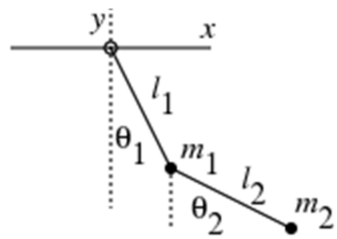
Consider a system consisting of a massless spring with spring constant k and a wheel with uniform mass M, moment of inertia I and radius R. The top of the wheel is connected to the end of the spring as illustrated in the Figure. The equilibrium length of the spring is at $x=0$. The wheel rolls without slipping on the horizontal plane when set in motion by the spring.

(a) Determine the kinetic energy T of the system.
(b) Determine the potential energy U of the system.
(c) Write down the equation describing the constraint on the motion of the wheel.
(d) Write down the Lagrangian L of the system.
(e) Derive Lagrange's equation of motion.
(f) What is the frequency of oscillations of the wheel?



I-3 [3,5,2]

Consider the motion of a coplanar, double pendulum system with one single pendulum hanging from another. The lengths of the massless strings, $\ell_1,\ \ell_2$ and the masses of the bobs, $m_1,\ m_2$ are different. Use the angle each string makes with the vertical as generalized coordinates, $\vartheta_1,\ \vartheta_2$.



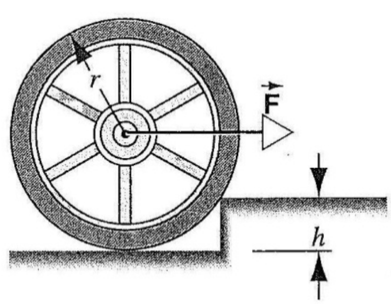
(a) Determine the Lagrangian equations of motion for *small oscillations*.

(b) Determine the normal modes of oscillation of the system and their corresponding frequencies. What are the frequencies for the special case of equal lengths and equal masses?

(c) Under what conditions will the system move as a single piece? Is this physically possible for a double pendulum?

I-4 [10]

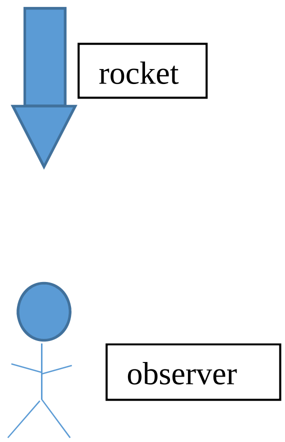
What minimum force *F*, applied horizontally at the axel of the wheel, is necessary to raise the wheel over a curb of height *h*? (See figure below.) The radius of the wheel is *r* and its total mass is *M*. The gravitational acceleration is *g*. You must express your answer in terms of *h*, *r*, *M*, and *g*.



I-5 [10]

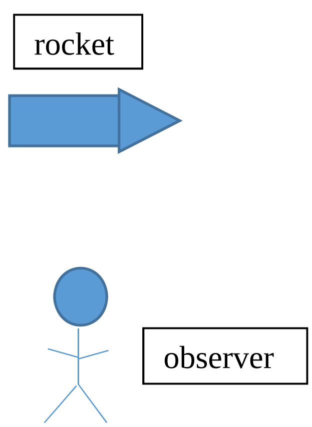
In the laboratory reference frame, an observer "sees" simultaneous bright flashes (the light flashes arrive at the observer at the same instant) from both the front and rear of a rocket of length L (as measured by the observer) which is travelling at a relativistic velocity, v.

If the rocket was travelling directly toward the observer in the laboratory frame, as shown below:



(a) Were the light pulses simultaneously emitted in the rocket frame?
(b) If not, which occurred first and what was the delay between emitted pulses?

If the rocket was travelling tangent to the observer in the laboratory frame, as shown below:



(c) Were the light pulses simultaneously emitted in the rocket frame?
(d) If not, which occurred first and what was the delay between emitted pulses?

I-6 [10]

Determine the electric field inside and outside a sphere of radius R and dielectric constant ϵ placed in a uniform electric field of magnitude E_0 directed along the z-axis.

I-7 [10]

(a) Write down Maxwell's equations for free space where there are no current or charge distributions.

(b) Derive wave equations for electric and magnetic fields from Maxwell's equations.

(c) Assume plane-wave solution for the fields,

$$\vec{E} = \overrightarrow{E_0} \cos\left(\vec{k} \cdot \vec{r} - \omega t\right)$$
$$\vec{B} = \overrightarrow{B_0} \cos\left(\vec{k} \cdot \vec{r} - \omega t\right) \quad ,$$
$$c = \omega / k$$

and show that $\vec{E}$, $\vec{B}$ and $\vec{k}$ are perpendicular to each other.

(d) Use the two curl equations in Maxwell's equations to show the relationship between ε_0, μ_0 and speed-of-light in free space, c.

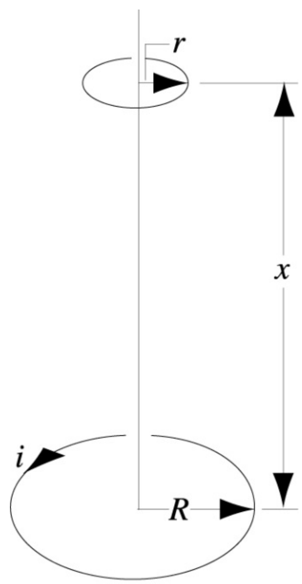
I-8 [10]

Consider two parallel loops of wire having a common axis as illustrated in the Figure. The smaller loop with radius r is above the larger loop with radius R by a distance $x >> R$. Consequently, the magnetic field $\boldsymbol{B}$, due the current i in the larger loop, is nearly constant throughout the smaller loop and equal to the value on the axis. Suppose that the distance x is increasing at the constant rate $dx/dt=v$.

(a) Determine the magnetic flux across the area bounded by the smaller loop as a function of x.

(b) Calculate the electromotive force ε in the smaller loop.

(c) Determine the direction of the induced current i_{ind} flowing in the smaller loop.



I-9 [10]

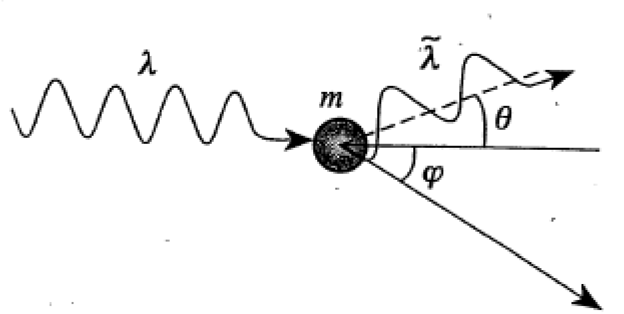
A massive atom with an atomic polarizability $\alpha(\omega)$ is subjected to an electromagnetic field (the atom being located at the origin),

$$\boldsymbol{E} = E_0 e^{i(kx-\omega t)}\hat{z} \ .$$

Find the asymptotic electric and magnetic fields radiated by the atom and calculate the energy radiated per unit solid angle. State any approximations used in this calculation, and state when (and why) they will break down as ω is increased.

I-10 [5,5]

In the Compton effect, a γ -ray photon of wavelength λ strikes a free but initially stationary, electron of mass m. The photon is scattered at an angle θ, and its scattered wavelength is $\tilde{\lambda}$. The electron recoils at an angle φ (see figure below).



(a) Write the relativistic equations for momentum and energy conservation.

(b) Find an expression for the change $\tilde{\lambda}$ - λ in the photon wavelength for the special case $\theta = \pi / 2$.

(I-1) No.

(a) From 2nd law:

$$F = -mk\left(v^4 + 2av^2 + a^2\right)$$

$$m\frac{dv}{dt} = -mk\left(v^4 + 2av^2 + a^2\right)$$

$$\cancel{m}v\frac{dv}{dx} = -\cancel{m}k\left(v^4 + 2av^2 + a^2\right)$$

$$\therefore \int_{v_0}^{v} \frac{v\,dv}{(v^2+a)^2} = -\int_{x_0}^{x} k\,dx$$

$$-\frac{1}{2}\left[\frac{1}{v^2+a}\right]_{v_0}^{v} = -\left[kx\right]_{x_0=0}^{x}$$

$$+\frac{1}{2k}\left[\frac{1}{v^2+a} - \frac{1}{v_0^2+a}\right] = +x$$

$$\therefore x = \frac{1}{2k}\left[\frac{1}{v^2+a} - \frac{1}{v_0^2+a}\right] \#$$

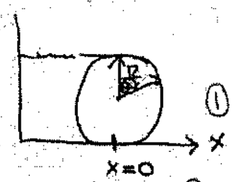
When $v=0$, i.e. "coming to a stop",

$$x = \frac{1}{2ka}\left[\frac{v_0^2}{v_0^2+a}\right] \#$$

(b) When $v_0 \to \infty$, the maximum distance become:

$$x_{max} = \left[\frac{1}{2ka}\right] \#$$

3



1

center of mass of wheel at x

rotation of wheel described by θ

(a) $T = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} I \dot{\theta}^2$ (1) (1)

(b) $U = \frac{1}{2} k (x - \cancel{x_0})^2$ (1)

(c) $V_{COM} = V_{tangential}$ for wheel when rolling without slipping

$$\dot{x} = R\dot{\theta} \quad (2)$$

(d) $L = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} I \dot{\theta}^2 - \frac{1}{2} k (x - \cancel{x_0})^2$ (1)

from constraint $\dot{\theta} = \dot{x}/R$

$$L = \frac{1}{2} m \dot{x}^2 + \frac{1}{2}\frac{I}{R^2} \dot{x}^2 - \frac{1}{2} k (x - \cancel{x_0})^2$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = m\ddot{x} + \frac{1}{2}\frac{I}{R^2}\ddot{x} \ (1) \qquad \frac{\partial L}{\partial x} = -k(x - \cancel{x_0}) \ (1)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0 \qquad \left(m + \frac{1}{2}\frac{I}{R^2}\right)\ddot{x} + k(x - \cancel{x_0}) = 0 \ (1)$$

$$\ddot{x} = -\underbrace{\frac{k}{(m + I/2R^2)}}_{\omega^2} x \quad (1)$$

Problem (I-3) Solution

(a) Let m_1, m_2 be the masses of the bobs and l_1, l_2 the lengths of the two strings as given in the figure. The two bobs have coordinates

$$(l_1 \sin\theta_1, -l_1\cos\theta_1)$$

$$(l_1 \sin\theta_1 + l_2 \sin\theta_2, -l_1\cos\theta_1 - l_2\cos\theta_2)$$

and velocities

$$(l_1\dot\theta_1\cos\theta_1, l_2\dot\theta_1 \sin\theta_1)$$

$$(l_2\dot\theta_1\cos\theta_1 + l_2\dot\theta_2\cos\theta_2, l_1\dot\theta_1\sin\theta_1 + l_2\dot\theta_2\sin\theta_2)$$

respectively. Then the K.E. T of the system is given by

$$2T = m_1 l_1^2\dot\theta_1^2 + m_2[l_1^2\dot\theta_2^2 + l_2^2\dot\theta_2^2 + 2l_1 l_2\dot\theta_1\dot\theta_2\cos(\theta_2-\theta_1)],$$

$$2T \approx (m_1+m_2)l_1^2\dot\theta_1^2 + m_2 l_2^2\dot\theta_2^2 + 2m_2 l_1 l_2\dot\theta_1\dot\theta_2$$

and potential energy V is given by

$$2V = -2m_1 g l_1\cos\theta_1 - 2m_2 g(l_1\cos\theta_1 + l_2\cos\theta_2)$$

$$\approx -2(m_1+m_2)gl_1(1-\tfrac{1}{2}\theta_1^2) - m_2 g l_2(1-\tfrac{1}{2}\theta_2^2)$$

$$= V_0 + (m_1+m_2)gl_1\theta_1^2 + m_2 g l_2\theta_2^2.$$

For small oscillations, we have retained only terms of up to order two of the small quantities $\theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2$.
Hence, the Lagrangian is $L = T - V$.

b.) To find the normal modes we write these in matrix form

$$2T = \sum_{i,j=1}^{2} M_{ij} \dot{\theta}_i \dot{\theta}_j = \dot{\Theta}' M \dot{\Theta}$$

$$2V = V_0 + \sum_{i,j=1}^{2} K_{ij} \theta_i \theta_j = V_0 + \Theta' K \Theta$$

with $M = \begin{pmatrix} (m_1+m_2) l_1^2 & m_2 l_1 l_2 \\ m_2 l_1 l_2 & m_2 l_2^2 \end{pmatrix}$

$$K = \begin{pmatrix} (m_1+m_2) g l_1 & 0 \\ 0 & m_2 g l_2 \end{pmatrix}$$

$$\Theta = \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}, \quad \dot{\Theta} = \begin{pmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{pmatrix}$$

and Θ', $\dot{\Theta}'$ being the transpose matrices of Θ, $\dot{\Theta}$ respectively.

Considering solutions of the type

$$\begin{pmatrix}\theta_1\\ \theta_2\end{pmatrix} = \begin{pmatrix}A_1\\ A_2\end{pmatrix}\cos(\omega t + \varepsilon)$$

we have $\quad (\mathbf{K} - \omega^2 \mathbf{M})\mathbf{A} = 0$

that is

$$\begin{pmatrix} (m_1+m_2)l_1(g - l_1\omega^2) & -m_2 l_1 l_2 \omega^2 \\ -m_2 l_1 l_2 \omega^2 & m_2 l_2 (g - l_2\omega^2) \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = 0$$

for $\mathbf{A}_1, A_2$ not to be zero identically

we require

$$\begin{vmatrix} (m_1+m_2)l_1(g - l_1\omega^2) & -m_2 l_1 l_2 \omega^2 \\ -m_2 l_1 l_2 \omega^2 & m_2 l_2 (g - l_2\omega^2) \end{vmatrix} = 0$$

its positive roots

$$\omega_{\pm} = \left\{ \frac{g}{2 l_1 l_2 m_1} \left[(m_1+m_2)(l_1+l_2) \pm \sqrt{(m_1+m_2)\left[m_2(l_1+l_2)^2 + m_1(l_1-l_2)^2\right]} \right] \right\}^{1/2}$$

are the ~~modes~~ normal mode angular frequencies of the system. As

$$\frac{A_1}{A_2} = \frac{l_2}{l_1}\left(\frac{g}{l_2\omega^2} - 1\right) \equiv$$

$$\equiv \frac{1}{2l_1}\left\{(l_1 - l_2) \mp \sqrt{\frac{m_2(l_1+l_2)^2 + m_1(l_1-l_2)^2}{(m_1+m_2)}}\right.$$

the normal modes are given by

$$\begin{pmatrix}\theta_1 \\ \theta_2\end{pmatrix} = \begin{pmatrix}\frac{l_1 - l_2}{2l_1} \mp \frac{1}{2l_1}\sqrt{(m_2(l_1+l_2)^2 + m_1(l_1-l_2)^2)/(m_1+m_2)} \\ 1\end{pmatrix} *$$

$$* A_\pm \cos(\omega_\pm t + \varepsilon_\pm)$$

where top and bottom signs correspond to ω_+, and ω_- respectively.

The general solution is:

$$\theta_1 = \left\{\frac{l_1 - l_2}{2l_1} - \frac{1}{2l_1}\sqrt{\frac{m_2(l_1+l_2)^2 + m_1(l_1-l_2)^2}{(m_1+m_2)}}\right\} A_+ \cos(\omega_+ t + \varepsilon_+)$$

$$+ \left\{\frac{l_1 - l_2}{2l_1} + \frac{1}{2l_1}\sqrt{\frac{m_2(l_1+l_2)^2 + m_1(l_1-l_2)^2}{(m_1+m_2)}}\right\} A_- \cos(\omega_- t + \varepsilon_-)$$

$$\theta_2 = A_+ \cos(\omega_+ t + \varepsilon_+) + A_- \cos(\omega_- t + \varepsilon_-)$$

where $A_+, A_-, \varepsilon_+, \varepsilon_-$ are constants to be determined from the initial conditions

For the special case of equal masses and equal lengths ($m_1 = m_2 = m$, $l_1 = l_2 = l$), the normal frequencies are

$$\omega_\pm = \sqrt{\frac{g}{l}\left(2 \pm \sqrt{2}\right)}$$

(c) For the system to move as a single piece, we require $\theta_1 = \theta_2$, that is

$$\frac{1}{2l_1}\left[(l_1 - l_2) \mp \sqrt{\frac{m_2(l_1+l_2)^2 + m_1(l_1-l_2)^2}{(m_1+m_2)}}\right] = 1$$

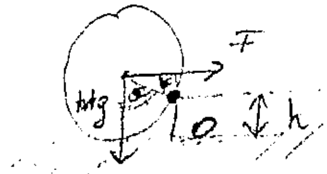
or

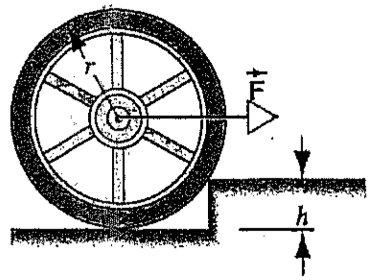
$$(m_1+m_2)(l_1+l_2) = \mp\sqrt{(m_1+m_2)^2(l_1+l_2)^2 - 4m_1(m_1+m_2)l_1 l_2}$$

as the left side is positive, the bottom sign of the right-hand side has to be used. Now squaring both sides gives $l_1 l_2 m_1 (m_1 + m_2) = 0$. This requires either $l_1 = 0$, or $l_2 = 0$, or $m_1 = 0$. Each of these cases will reduce the two pendulum system into a one pendulum system. Hence, the two-pendulum system cannot move as a single piece.

I-4

I-4 [10]
What minimum force F, applied horizontally at the axel of the wheel, is necessary to raise the wheel over a curb of height h? (See figure below.) The radius of the wheel is r and its total mass is M. The gravitational acceleration is g. You must express your answer in terms of h, r, M, and g.





Torques from gravity and force F just balance / equal at the critical value.

Torques about point O: $\vec{r} \times M\vec{g} + \vec{r} \times \vec{F} = 0$

$$Mg\cancel{r}\sin\theta = F\cancel{r}\cos\theta$$

$$F = Mg\frac{\sin\theta}{\cos\theta} = Mg\tan\theta$$ (Must express $\tan\theta$ with r and h)

$$h + r\cos\theta = r \qquad \cos\theta = \frac{r-h}{r} = 1 - \frac{h}{r}$$

$$\sin\theta = \sqrt{1-\cos^2\theta} = \sqrt{1-\left(1-\frac{h}{r}\right)^2} = \sqrt{2\frac{h}{r} - \left(\frac{h}{r}\right)^2}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\sqrt{2\frac{h}{r} - \left(\frac{h}{r}\right)^2}}{1-\frac{h}{r}} = \frac{\sqrt{2hr-h^2}}{r-h} = \frac{\sqrt{h(2r-h)}}{r-h}$$

$$\boxed{F = Mg\frac{\sqrt{h(2r-h)}}{r-h}}$$

$(r > h)$

I-5solution:

(a) and (c) in neither case are the events simultaneous in the reference frame of the train.

(b) If the observer "sees" light from both the front and rear of the train simultaneously, then the time delay between events as measured from the observer is $\delta t = L/c$.
Performing a Lorentz transformation to arrive at $\delta t'$ in the train reference frame yields,

$$\delta t' = \gamma\left(\delta t - \frac{vL}{c^2}\right) = \left(\frac{L}{c} - \frac{vL}{c^2}\right)\gamma = \frac{L\left(1 - \frac{v}{c}\right)}{\sqrt{c^2 - v^2}}$$

(d) If the observer "sees" light from both the front and rear, while the train is moving tangent to the observer then both light pulses are viewed to be emitted simultaneously by the observer, so $\delta t = 0$. Lorentz transformation then yields.

$$\delta t' = \frac{\gamma v L}{c^2} = \frac{v}{c}\frac{L}{\sqrt{c^2 - v^2}}$$

I-6 solution:

Let the origin be at the spherical center and take the direction of the original field E to define the polar axis z. Let the electrostatic potential at a point inside the sphere be Φ_1, and the potential at a point outside the sphere be Φ_2. By symmetry we can write Φ_1 and Φ_2 as

$$\Phi_1 = \sum_{n=0} (A_n r^n + B_n/r^{n+1})P_n(cos\theta),$$

$$\Phi_2 = \sum_{n=0} (C_n r^n + D_n/r^{n+1})P_n(cos\theta)$$

Where P_n are Legendre polynomials. The boundary conditions are as follows:

(1) Φ_1 is finite at $r = 0$.
(2) $\Phi_2\ (r \to \infty) = -Ercos\theta = -ErP_1(cos\theta)$
(3) $\Phi_1 = \Phi_2$ (at $r = a$), $\epsilon \frac{\partial \Phi_1}{\partial r} = \epsilon_0 \frac{\partial \Phi_2}{\partial r}$ (at $r = a$)

From conditions (1) and (2), we obtain

$$B_n = 0, \quad C_1 = -E, \quad C_n = 0 \ (n \neq 1).$$

The from condition (3), we obtain

$$-EaP_1(cos\theta) + \sum_n \frac{D_n}{a^{n+1}} P_n(cos\theta) = \sum_n A_n a^n P_n(cos\theta),$$

$$-\epsilon\left(EP_1(cos\theta) + \sum_n (n+1)\frac{D_n}{a^{n+2}} P_n(cos\theta)\right) = \epsilon_0 \sum_n nA_n a^{n-1} P_n(cos\theta)$$

In order to satisfy for all θ, the coefficients of $P_n(cos\theta)$ must be equal term-by-term for each n. This gives,

$$A_1 = -\frac{3\epsilon E}{\epsilon_0 + 2\epsilon}, \quad D_1 = \frac{\epsilon_0 - \epsilon}{\epsilon_0 + 2\epsilon} Ea^3, \qquad A_n = D_n = 0 \ (n \neq 1)$$

yielding,

$$\Phi_1 = -\frac{3\epsilon E}{\epsilon_0 + 2\epsilon} rcos\theta$$

$$\Phi_2 = -\left(1 - \frac{\epsilon_0 - \epsilon}{\epsilon_0 + 2\epsilon}\left(\frac{a}{r}\right)^3\right) Ercos\theta$$

(I-7)

No.

(a) Maxwell's Equ

$\nabla \cdot \vec{E} = 0$

$\nabla \cdot \vec{B} = 0$

$\nabla \times E = -\frac{\partial B}{\partial t}$

$\nabla \times B = \mu_0 \varepsilon_0 \frac{\partial E}{\partial t}$ #

(b) Wave Equ

$\nabla \times (\nabla \times \vec{E}) = -\frac{\partial}{\partial t}(\nabla \times \vec{B})$

$LHS = \nabla \times (\nabla \times E) = \nabla(\cancel{\nabla \cdot \vec{E}}^{\,0}) - \nabla^2 E$

$RHS = -\frac{\partial}{\partial t}(\nabla \times \vec{B}) = -\varepsilon_0 \mu_0 \frac{\partial^2}{\partial t^2} E$

$\therefore \nabla^2 \vec{E} = \varepsilon_0 \mu_0 \frac{\partial^2}{\partial t^2} \vec{E}$

Similarly, $\nabla^2 \vec{B} = \varepsilon_0 \mu_0 \frac{\partial^2}{\partial t^2} \vec{B}$ #

Wave Equ. in free-space

(c) Directionality

$\nabla \cdot \vec{E} = \vec{k} \cdot \vec{E} = 0$, $\therefore \vec{k} \perp \vec{E}$

$\nabla \cdot \vec{B} = \vec{k} \cdot \vec{B} = 0$ $\therefore \vec{k} \perp \vec{B}$

$\nabla \times \vec{E} = -\frac{\partial B}{\partial t}$, $\therefore \vec{k} \times \vec{E} = \omega \vec{B}$, $\therefore \vec{E} \perp \vec{B}$ #

(d) speed of Light.

from Wave equ. $\nabla^2 E = \varepsilon_0 \mu_0 \frac{\partial^2}{\partial t^2} E$

$k^2 = \varepsilon_0 \mu_0 \omega^2$

$\frac{w^2}{k^2} = \frac{1}{\varepsilon_0 \mu_0} = c^2$

$\therefore c = \sqrt{1/\varepsilon_0 \mu_0}$ #

Zeitsoon

The angle ϕ between the current element $i\,d\vec{s}$ and $\vec{r}$ is 90°. From the Biot–Savart law, we know that the vector $d\vec{B}$ for this element is at right angles to the plane formed by $i\,d\vec{s}$ and $\vec{r}$ and thus lies at right angles to $\vec{r}$, as the figure shows.

Let us resolve $d\vec{B}$ into two components; one, $d\vec{B}_z$, along the axis of the loop and another, $d\vec{B}_\perp$, at right angles to the axis. Only $d\vec{B}_z$ contributes to the total magnetic field $\vec{B}$ at point P. This follows because the components $d\vec{B}_z$ for all current elements lie on the axis and add directly; however, the components $d\vec{B}_\perp$ point in different directions perpendicular to the axis, and the sum of all $d\vec{B}_\perp$ for the complete loop is zero, from symmetry. (A diametrically opposite current element, indicated in Fig. 33-10, produces the same $d\vec{B}_z$ but $d\vec{B}_\perp$ in the opposite direction.) We can therefore replace the vector integral over all $d\vec{B}$ with an integral over the z components only, and the magnitude of the field is given by

$$B = \int dB_z. \qquad (33\text{-}14)$$

For the current element in Fig. 33-10, the Biot–Savart law (Eq. 33-9) gives

$$dB = \frac{\mu_0 i}{4\pi}\frac{ds \sin 90^\circ}{r^2}. \qquad (33\text{-}15)$$

We also have

$$dB_z = dB \cos\alpha,$$

which, combined with Eq. 33-15, gives

$$dB_z = \frac{\mu_0 i \cos\alpha \, ds}{4\pi r^2}. \qquad (33\text{-}16)$$

Figure 33-10 shows that r and α are not independent of each other. Let us express each in terms of z, the distance from the center of the loop to the point P. The relationships are

$$r = \sqrt{R^2 + z^2}$$

and

$$\cos\alpha = \frac{R}{r} = \frac{R}{\sqrt{R^2 + z^2}}.$$

Substituting these values into Eq. 33-16 for dB_z gives

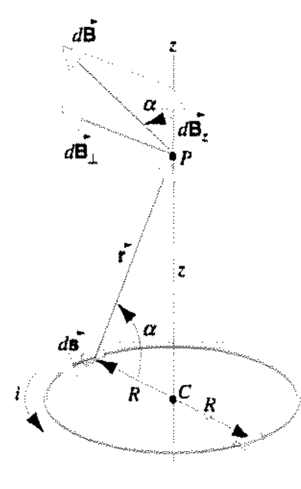
$$dB_z = \frac{\mu_0 i R}{4\pi(R^2 + z^2)^{3/2}}\,ds. \qquad (33\text{-}17)$$

Note that i, R, and z have the same values for all current elements. Integrating this equation, we obtain

$$B = \int dB_z = \frac{\mu_0 i R}{4\pi(R^2 + z^2)^{3/2}} \int ds \qquad (33\text{-}18)$$

or, noting that $\int ds$ is simply the circumference of the loop $(= 2\pi R)$,

$$B = \frac{\mu_0 i R^2}{2(R^2 + z^2)^{3/2}}. \qquad (33\text{-}19)$$



I-8 (Cont'd)

P34-6 (a) Far from the plane of the large loop we can approximate the large loop as a dipole, and then

$$B = \frac{\mu_0 i \pi R^2}{2x^3}.$$

The flux through the small loop is then

$$\Phi_B = \pi r^2 B = \frac{\mu_0 i \pi^2 r^2 R^2}{2x^3}.$$

(b) $\mathcal{E} = -d\Phi_B/dt$, so

$$\mathcal{E} = \frac{3\mu_0 i \pi^2 r^2 R^2}{2x^4} v.$$

(c) Anti-clockwise when viewed from above.

I-9 solution:

The atom acts as a Hertzian dipole at the origin with dipole moment $\boldsymbol{P} = \alpha\boldsymbol{E} = \alpha E_0 e^{-i\omega t}\hat{z}$. At large r the asymptotic (radiation) electric and magnetic fields radiated by the atom are

$$B(r,t) = -\frac{\alpha E_0 \omega^2}{4\pi\epsilon_0 c^3 r} sin\theta e^{-i\omega t}\hat{\phi}$$

$$E(r,t) = -\frac{\alpha E_0 \omega^2}{4\pi\epsilon_0 c^3 r} sin\theta e^{-i\omega t}\hat{\theta}$$

The energy radiated per unit solid angle is

$$\frac{dW}{d\Omega} = \frac{<N>}{r^{-2}} = \frac{c}{2\mu_0 r^{-2}}|B|^2 = \frac{\alpha^2 E_0^2 \omega^4}{32\pi^2\epsilon_0 c^3}\sin^2\theta$$

The approximation used is $r \gg \lambda \gg l$, where l is the linear dimension of the atom and $\lambda = 2\pi c/\omega$. As ω is increased, λ will decrease and eventually become smaller than l, thus invalidating the approximation.

QE-Ph.D.

[I-10] **Solution:**

(a.) From momentum and energy conservation we may write

$$\vec{p} = \tilde{p} + \vec{p}_e$$

$$\mathcal{E} + \mathcal{E}_e = \tilde{\mathcal{E}} + \tilde{\mathcal{E}}_e$$

where $\vec{p}, \tilde{p}, \mathcal{E}, \tilde{\mathcal{E}}$ are the momenta and energies of the photon before and after the scattering, respectively, $\vec{p}_e$, $\tilde{\mathcal{E}}_e$ are the final momentum and energies of the electron, and $\mathcal{E}_e$ is its initial energy. We have for the electron

$$\tilde{\mathcal{E}}_e = \sqrt{p_e^2 c^2 + m^2 c^4}\,, \quad \mathcal{E}_e = mc^2$$

and for the photon

$$\mathcal{E} = pc\,, \quad \tilde{\mathcal{E}} = \tilde{p}c$$ we may now

rewrite the above equations in the form

$$\vec{p} - \tilde{p} = \vec{p}_e$$

$$pc + mc^2 = \tilde{p}c + \sqrt{p_e^2 c^2 + m^2 c^4}$$

(b.) To solve these equations we may express the momentum of the recoil electron, p_e, in two ways:

[I-10] continued.

$$\vec{p}_e^{\,2} = (\vec{p} - \tilde{\vec{p}})^2$$

$$p_e^2 = (p - \tilde{p})^2 + 2mc(p - \tilde{p})$$ using the respective equations from part (a.)

$$p\tilde{p}(1 - \cos\theta) = mc(p - \tilde{p})$$

and when $\theta = \pi/2$ $\cos\theta = 0$. We have

$$p\tilde{p} = mc(p - \tilde{p})$$ or we may divide by $p\tilde{p}$ and obtain

$$1 - mc\left(\frac{1}{\tilde{p}} - \frac{1}{p}\right)$$ but $p = h/\lambda$

then the final result becomes

$$\tilde{\lambda} - \lambda = \frac{h}{mc} \, .$$

Physics PhD Qualifying Examination
Part II – Friday, August 28, 2015

Name:____
(please print)
Identification Number:_________

STUDENT: insert a check mark in the left boxes to designate the problem numbers that you are handing in for grading.
PROCTOR: check off the right hand boxes corresponding to the problems received from each student. Initial in the right hand box.

	1	
	2	
	3	
	4	
	5	
	6	
	7	
	8	
	9	
	10	

Student's initials

\# problems handed in:

Proctor's initials

INSTRUCTIONS FOR SUBMITTING ANSWER SHEETS

1. DO NOT PUT YOUR NAME ON ANY ANSWER SHEET. EXAMS WILL BE COLLATED AND GRADED BY THE ID NUMBER ABOVE.
2. Use at least one separate preprinted answer sheet for each problem. Write on only one side of each answer sheet.
3. Write your **identification number** listed above, in the appropriate box on the preprinted sheets.
4. Write the **problem number** in the appropriate box of each preprinted answer sheet. If you use more than one page for an answer, then number the answer sheets with both problem number and page (e.g. Problem 9 – Page 1 of 3).
5. Staple together all the pages pertaining to a given problem. Use a paper clip to group together all problems that you are handing in.
6. **There is no limit on the number of problems you can turn in.**
7. A passing distribution for the individual components will normally include at least four passed problems (from problems 1-6) for Quantum Physics and two problems (from problems 7-10) for Thermodynamics and Statistical Mechanics.
8. **YOU MUST SHOW ALL YOUR WORK.**

II-1 [10]

A particle of mass m moves in one dimension under the influence of a potential $V(x)$. Suppose it is an energy eigenstate $\psi(x) = \left(\frac{\gamma^2}{\pi}\right)^{1/4} e^{-\gamma^2 x^2/2}$ with energy $E = \hbar^2\gamma^2/2m$.

(a) Find the mean position of the particle.
(b) Find the mean momentum of the particle.
(c) Find $V(x)$.
(d) Find the probability $P(p)dp$ that the particle's momentum is between p and $p + dp$.

II-2 [10]

A weak perturbative potential is added to the *one-dimensional simple harmonic oscillator* with mass m and frequency ω in the form of

$$V = \lambda x\,,$$

where λ is a constant.

(a) Find the *lowest-order non-vanishing corrections* to *all* energy levels and write down the new perturbed energy levels to this order.

(b) Obtain the *lowest-order non-vanishing corrections* to *all* eigenkets of the unperturbed Hamiltonian and write down the new perturbed eigenkets to this order.

You may find the number representation of the harmonic oscillator with the annihilation and creation operators useful

$$a = \sqrt{\frac{m\omega}{2\hbar}}\left(x + \frac{i}{m\omega}p\right), \qquad a^{+} = \sqrt{\frac{m\omega}{2\hbar}}\left(x - \frac{i}{m\omega}p\right).$$

II-3 [10]

(a) Consider a one-electron atom, write done its spin-orbit Hamiltonian.

(b) Let us assume that the un-perturbed atomic wavefunction of the atom (i.e. in the absence of spin-orbit interaction) is given by $|\psi\rangle$. Now, when the spin-orbit interaction is present, use perturbation theory to find the eigenvalue of the state $|\psi\rangle$.

(c) Describe the degeneracy of the states.

(d) Work out the degeneracy for state with L=1 and S=1/2.

II-4 [10]

In the Born approximation, evaluate the cross-section of scattering by a "delta-function" potential. The scattering potential is given by: $V(\vec{r}) = B\delta(\vec{r})$, where we take the force center as the origin.

(a) Obtain the differential cross-section and the total scattering cross section using the Born approximation.

(b) Discuss the case with a delta-function potential with respect to the interaction-potential of very small range (r_{eff}) which is much less than the deBroglie wavelength. In this scattering isotropic and thus velocity independent? Discuss the applicability of Born approximation.

Note that *B* is a constant and equals to the volume integral of the delta-function potential.

II-5 [10]

The relationship between the wavelength λ and frequency ν for the propagation of electromagnetic waves through a hollow waveguide is $\lambda = \frac{c}{\sqrt{\nu^2 - \nu_0^2}}$ with c being the speed of light and ν_0 standing for the minimum frequency for which waves will propagate.
Calculate the group velocity v_g and phase velocity v_{ph} of the waves.

II-6 [6,4]

A hydrogen atom in its *ground state* is placed between the parallel plates of a capacitor. For times $t < 0$, no voltage is applied. Starting at $t = 0$, an electric field $\boldsymbol{E}(t) = E_o \hat{\boldsymbol{z}} e^{-t/\tau}$ is applied, where τ is a constant.

(a) Derive the equation for the probability that the electron ends up in a state j due to this perturbation**.**

(b) Evaluate the result if state j is a:

(*i*) 2s state (parity argument may simplify the calculation);
(*ii*) 2p state.

The normalized eigenstates of the hydrogen atom (you may not need all):

$$\varphi_{100} = \frac{1}{\sqrt{\pi} a_0^{3/2}} e^{-r/a_0} \ ,$$

$$\varphi_{200} = \frac{1}{(2a_0)^{3/2}\sqrt{\pi}} \left(1 - \frac{r}{2a_0}\right) e^{-r/2a_0} \ ,$$

$$\varphi_{210} = \frac{1}{(2a_0)^{3/2}\sqrt{\pi}} \left(\frac{r}{2a_0}\right) e^{-r/2a_0} \cos\theta , \qquad \varphi_{21\pm1} = \frac{1}{8a_0^{3/2}} \left(\frac{r}{a_0}\right) e^{-r/2a_0} \sin\theta \, e^{\pm\phi} \ ,$$

where a_0 is the Bohr radius.

II-7 [3,7]

The Van der Waals equation of state is given by

$$\left(P + a\left(\frac{N}{V}\right)^2\right)(V - bN) = NkT \ ,$$

where P is the pressure, V is the volume, T is the temperature, N is the number of atoms, and k is the Boltzmann constant. a and b are material specific constants.

(a) Give a physical interpretation of the above equation, in particular, describe the role of the constants a and b.
(b) Express the critical temperature, volume, and pressure, T_c , V_c , and P_c , in terms of the constants a, b, k, and N.

II-8 [2,4,4]

The Helmholtz free energy of a dilute plasma gas consisting of N electrons confined to a volume V at temperature T is given by

$$F(T,V,N) = F^{ideal}(T,V,N) - \frac{2}{3}Ne^2\left(\frac{4\pi Ne^2}{Vk_BT}\right)^{1/2},$$

where e is the electron charge, k_B is Boltzmann's constant, and $F^{ideal}(T,V,N)$ is the Helmholtz free energy of a mono-atomic ideal gas.

(a) Obtain the equation of state of the above plasma gas.
(b) Obtain the internal energy of the system $E(T,V,N)$.
(c) Obtain the constant-volume heat capacity C_V .

Note: your answers should be fully explicit in terms of the variables T , V , and N . To that end, you are expected to remember and use the equation of state, internal energy, heat capacity of the ideal gas in obtaining your final results.

II-9 [10]

Two containers are at the same temperature *T*. The first contains gas at pressure p_1 whose molecules have mass m_1 with a root-mean-square speed v_{rms1}. The second contains molecules of mass m_2 at pressure $2p_1$ that have an average speed $v_{av2}=2v_{rms1}$. Find the ratio m_1/m_2 of the masses of the molecules.

II-10 [10]

Consider a *two-dimensional photon gas* confined to an area $A = L \times L$. What is the average number of photons in the system at temperature T?
Your answer must be expressed in terms of *A*, *T*, and of course, the necessary fundamental constants.

To keep your expressions relatively compact, the following relationship will come useful:
$\int_0^\infty \frac{x^{\nu-1}}{e^x - 1} dx = \Gamma(\nu)\zeta(\nu)$, where $\Gamma(\nu)$ is the gamma function and $\zeta(\nu)$ is the Rieman zeta function. In particular, $\zeta(2) = \frac{\pi^2}{6}$.
(You must *derive* your answer. Guessing or hand-waving the answer, or pulling it out of your memory, will yield zero credit.)

Solution QE-II Aug 2015

II-1 solution:

(a) The mean position of the particle is

$< x >= \int_{-\infty}^{\infty} \psi^*(x) x \psi(x) dx = \gamma/\sqrt{\pi} \int_{-\infty}^{\infty} x e^{-\gamma^2 x^2} dx = 0$

(b) The mean momentum is

$< p >= \int_{-\infty}^{\infty} \psi^*(x) \frac{\hbar}{i} (\frac{d}{dx} \psi(x)) dx = \gamma\hbar/i\sqrt{\pi} \int_{-\infty}^{\infty} x e^{-\frac{\gamma^2 x^2}{2}} \frac{d}{dx} (e^{-\gamma^2 x^2/2}) dx = 0$

(c) From the SE eqn,

$$-\frac{\hbar^2}{2m} \frac{d^2}{dx^2} \psi(x) = (E - V(x))\psi(x)$$

As

$$-\frac{\hbar^2}{2m} \frac{d^2}{dx^2} e^{-\frac{\gamma^2 x^2}{2}} = -\frac{\hbar^2}{2m} (-\gamma^2 + \gamma^4 x^2) e^{-\frac{\gamma^2 x^2}{2}}$$

we have,

$$E - V(x) = -\frac{\hbar^2}{2m} (-\gamma^2 + \gamma^4 x^2)$$

or

$$V(x) = -\frac{\hbar^2}{2m} (\gamma^4 x^2 - \gamma^2) + \frac{\hbar^2 \gamma^2}{2m} = \frac{\hbar^2 \gamma^4 x^2}{2m}.$$

(d) $\psi(p)$ can be obtained by direct Fourier transform,

$$\psi(p) = \int \frac{dx}{\sqrt{2\pi\hbar}} e^{-\frac{ipx}{\hbar}} \left(\frac{\gamma^2}{\pi}\right)^{1/4} e^{-\gamma^2 x^2/2} = \int \frac{dx}{\sqrt{2\pi\hbar}} e^{-\frac{ip^2}{2\hbar^2\gamma^2}} \left(\frac{\gamma^2}{\pi}\right)^{1/4} e^{\left(\frac{p}{\sqrt{2}\hbar\gamma} - \frac{i\gamma x}{\sqrt{2}}\right)^2}$$
$$= \left(\frac{1}{\hbar^2\gamma^2\pi}\right)^{1/4} e^{-p^2/2\hbar^2\gamma^2}$$

and,

$$P(p)dp = \psi(p)\psi^*(p)dp = \frac{1}{\hbar\gamma\sqrt{\pi}} e^{-p^2/\hbar^2\gamma^2} dp$$

II-2

$H = H_0 + V \qquad H_0 = \frac{p^2}{2m} + \frac{m\omega^2}{2} x^2 \; ; \; V = \lambda X$

$H_0 |n^{(0)}\rangle = E_n^{(0)} |n^{(0)}\rangle \qquad$ SHO

$E_n^{(0)} = \hbar\omega (1/2 + n) \qquad n = 0, 1, 2, \ldots$ (non-degenerate)

recall: $X = \sqrt{\frac{\hbar}{2m\omega}} (a + a^+)$

$\langle n^{(0)} | m^{(0)} \rangle = \delta_{nm} \; ; \quad a|n^{(0)}\rangle = \sqrt{n}\, |(n-1)^0\rangle$

$\qquad a^+ |n^{(0)}\rangle = \sqrt{n+1}\, |(n+1)^0\rangle$

(a) $o(\lambda)$ energy correction: (first order energy correction)

$$\Delta_n^{(1)} = \langle n^{(0)} | V | n^{(0)} \rangle = \lambda \langle n^{(0)} | X | n^{(0)} \rangle \equiv 0$$

Hence, must go to $o(\lambda^2)$ in energy correction:

$$\Delta_n^{(2)} = \sum_{k \neq n} \frac{|V_{nk}|^2}{E_n^{(0)} - E_k^{(0)}}$$

$$V_{nk} = \langle n^{(0)} | V | k^{(0)} \rangle = \lambda \langle n^{(0)} | X | k^{(0)} \rangle =$$

$$= \lambda \sqrt{\frac{\hbar}{2m\omega}} \left\{ \langle n^{(0)} | a | k^{(0)} \rangle + \langle n^{(0)} | a^+ | k^{(0)} \rangle \right\}$$

$$= \lambda \sqrt{\frac{\hbar}{2m\omega}} \left\{ \sqrt{k} \langle n^{(0)} | (k-1)^0 \rangle + \sqrt{(k+1)} \langle n^{(0)} | (k+1)^0 \rangle \right\}$$

-1- $\qquad = \lambda \sqrt{\frac{\hbar}{2m\omega}} \left\{ \sqrt{k}\, \delta_{n,k-1} + \sqrt{k+1}\, \delta_{n,k+1} \right\} =$

$$= \lambda\sqrt{\frac{\hbar}{2m\omega}}\left\{\sqrt{n+1}\,\delta_{k,n+1} + \sqrt{n}\,\delta_{k,n-1}\right\}$$

$$\Delta_n^{(2)} = \sum_{k\neq n}\frac{|V_{nk}|^2}{E_n^{(0)}-E_k^{(0)}} = \lambda^2\frac{\hbar}{2m\omega}\left(\frac{n+1}{E_n^{(0)}-E_{n+1}^{(0)}} + \frac{n}{E_n^{(0)}-E_{n-1}^{(0)}}\right)$$

$$E_n^{(0)}-E_{n+1}^{(0)} = \hbar\omega(1/2+n) - \hbar\omega(1/2+(n+1)) = -\hbar\omega$$

$$E_n^{(0)}-E_{n-1}^{(0)} = \hbar\omega(1/2+n) - \hbar\omega(1/2+(n-1)) = +\hbar\omega$$

$$\Delta_n^{(2)} = \frac{\lambda^2\hbar}{2m\omega}\left(\frac{n+1}{-\hbar\omega} + \frac{n}{\hbar\omega}\right) =$$

$$= \frac{\lambda^2\hbar}{2m\omega}\left(-\frac{1}{\hbar\omega}\right) = \boxed{-\frac{\lambda^2}{2m\omega^2}} \qquad \forall n$$

$$E_n^{(2)} = E_n^{(0)} + \Delta_n^{(2)} = \hbar\omega(1/2+n) - \frac{\lambda^2}{2m\omega^2}$$

(b)

$|n\rangle \simeq |n^{(0)}\rangle + |n^{(1)}\rangle$, where

$$|n^{(1)}\rangle = \sum_{k\neq n}\frac{V_{kn}}{E_n^{(0)}-E_k}|k^{(0)}\rangle$$

see V_{kn} above in (a)

So $o(\lambda)$ for perturbed eigenstates will suffice

II-2 (cont.)

$$|n^{(1)}\rangle = \lambda\sqrt{\frac{\hbar}{2m\omega}}\left(\frac{\sqrt{n+1}}{E_n^{(0)}-E_{n+1}^{(0)}}\,|(n+1)^{0}\rangle + \frac{\sqrt{n}}{E_n^{(0)}-E_{n-1}^{(0)}}\,|(n-1)^{0}\rangle\right)$$

$$= \lambda\sqrt{\frac{\hbar}{2m\omega}}\left(-\frac{\sqrt{n+1}}{\hbar\omega}|(n+1)^{0}\rangle + \frac{\sqrt{n}}{\hbar\omega}|(n-1)^{0}\rangle\right)$$

$$= \lambda\sqrt{\frac{1}{2m\hbar\omega^{3}}}\left(\sqrt{n}\,|(n-1)^{0}\rangle - \sqrt{n+1}\,|(n+1)^{0}\rangle\right)$$

So finally,

$$\boxed{|n\rangle \simeq |n^{(0)}\rangle + \lambda\sqrt{\frac{1}{2m\hbar\omega^{3}}}\left(\sqrt{n}\,|(n-1)^{0}\rangle - \sqrt{(n+1)}\,|(n+1)^{0}\rangle\right)}$$

(obviously, for $n=0$, no first term)

(normalization won't change $O(\lambda)$ as that is an $O(\lambda^2)$ effect)

(II-3)

No.

(a) Spin-orbit Hamiltonian

$$H = \xi \vec{L} \cdot \vec{S} \;\#$$

(b) Eigenvalues of the state $|\psi\rangle$.

Total Angular momentum: $J \overset{def.}{=} L + S$

$$\therefore \vec{L} \cdot \vec{S} = (J^2 - L^2 - S^2)/2$$

Since the unperturbed $|\psi\rangle$ is also eigenstate of J^2, L^2, S^2:

$$J^2|\psi\rangle = J(J+1)|\psi\rangle$$
$$L^2|\psi\rangle = L(L+1)|\psi\rangle$$
$$S^2|\psi\rangle = S(S+1)|\psi\rangle$$

$$\therefore \vec{L} \cdot \vec{S}|\psi\rangle = \frac{1}{2}\left(J(J+1) - L(L+1) - S(S+1)\right)|\psi\rangle \;\#$$

(c) The degeneracy is the degeneracy of $m_j = 2J+1$ #

(d) For $L=1$, $S=1/2$, $J=3/2$ with degeneracy of 4

and $J=1/2$ with degeneracy of 2 #

(a) Scattering potential

$$V(r) = B\,\delta(r), \text{ or}$$

$$B = \int V(r)\, d\vec{r} = \text{const} \quad (1)$$

Differential cross section into a unit solid-angle is:

Born Approxi:
$$\frac{d\sigma}{d\Omega} = \frac{\mu^2}{4\pi^2\hbar^2}\int V(\vec{r})\, e^{i\vec{q}\cdot\vec{r}}\, d^3\vec{r}, \quad (2)$$

- μ is the reduced mass
- $\hbar\vec{q} = \vec{p} - \vec{p}\,'$ (momentum change of the colliding particle)

$$\frac{d\sigma}{d\Omega} = \frac{\mu^2 B^2}{4\pi^2\hbar^4} \quad \# \quad (3)$$

The scattering by a delta-function potential is isotropic and does not depend on the velocity.

(In other words, as $\delta(r)$ is only relevant at $r \approx 0$, $\vec{q}\cdot\vec{r}$ becomes independent of $\vec{q} = \frac{1}{\hbar}(\vec{p} - \vec{p}\,')$)

The total cross-section, i.e. over a (4π) solid-angle

$$\sigma = (4\pi)\frac{d\sigma}{d\Omega} = \frac{\mu^2 B^2}{\pi \hbar^4} \quad \# \qquad (4)$$

By examining equation (2), we see

(b) the range of interaction (r_{eff}) must be sufficiently small such that
$$\begin{cases} \vec{q}\cdot r_{eff} \ll 1 \\ e^{i\vec{q}\cdot r_{eff}} \approx 1 . \end{cases}$$

In this case, $V(r)$ is only appreciable within "r_{eff}" and is zero everywhere else.

II-5

EXAMPLE 2.10 The relation between the wavelength λ and frequency ν for the propagation of electromagnetic waves through a wave guide (typically a hollow rectangular or cylindrical metal pipe) is given by

$$\lambda = \frac{c}{\sqrt{\nu^2 - \nu_0^2}}$$

What are the phase and group velocities of these waves? *Note*: For a wave guide the constant ν_0 is the minimum frequency for which the waves will propagate.

SOLUTION The relationship between the wavelength λ and the frequency ν can also be expressed in terms of the wave vector k and the angular frequency ω as

$$kc = \sqrt{\omega^2 - \omega_0^2}$$

or

$$\omega = \sqrt{(kc)^2 + \omega_0^2} = c\sqrt{k^2 + (\omega_0/c)^2}$$

where $\omega_0 = 2\pi\nu_0$. The phase velocity is given by

$$v_{\text{ph}} = \frac{\omega}{k} = \frac{c\sqrt{k^2 + (\omega_0/c)^2}}{k} = c\sqrt{1 + (\omega_0/kc)^2}$$

which is greater than c, while the group velocity is given by

$$v_{\text{g}} = \frac{d\omega}{dk} = c\frac{k}{\sqrt{k^2 + (\omega_0/c)^2}} = \frac{c}{\sqrt{1 + (\omega_0/kc)^2}}$$

which is less than c. Note that $v_{\text{ph}}v_{\text{g}} = c^2$.

A phase velocity that exceeds the speed of light may seem troubling at first. But it is the group velocity that determines how fast, say, information is transmitted by a localized wave packet. The phase velocity is simply the velocity of a particular point on a wave with a definite wavelength, a wave that extends throughout space.

Q.E.-Ph.D. August 2015 John Schroeder

[II-6] Solution

For time-dependent perturbations a general wave function is

$$\Psi(\vec{r},t) = \sum_j a_j(t)\,\psi_j(\vec{r})\,e^{-i\omega_j t}$$

where the ψ_j satisfy

$$H_0 \psi_j = \hbar\omega_j \psi_j$$

For the time dependent perturbation $V(t)$,

$$V(t) = -e|\vec{E}_0|\, z\, e^{-t/\tau}.$$

From Schrödinger's equation we can derive an equation for the time development of the amplitudes $a_j(t)$:

$$i\hbar \frac{\partial}{\partial t}\Psi = [H_0 + V(t)]\Psi$$

$$i\hbar \frac{\partial}{\partial t} a_j(t) = \sum_l a_l \langle j|V(t)|l\rangle e^{i(\omega_j - \omega_l)t}$$

If the system is originally in the ground state, we have $a_{1s}(0) = 1$ and the other values of $a_j(0)$ are zero. For small

[II-B] continued.

Perturbations it is sufficient to solve the equation for $j \neq 1s$:

$$\frac{\partial}{\partial t} a_j(t) = \frac{ie|\vec{E}_0|}{\hbar} \langle j|z|1s\rangle e^{-t\left\{\frac{1}{\tau} - i(\omega_j - \omega_{1s})\right\}}$$

$$a_j(\infty) = \frac{ie|\vec{E}_0|\langle z\rangle}{\hbar}\int_0^\infty dt\, e^{-t\left\{\frac{1}{\tau} - i(\omega_j - \omega_{1s})\right\}}$$

$$a_j(\infty) = \frac{ie|\vec{E}_0|\langle z\rangle \tau}{\hbar[1 - i\tau(\omega_f - \omega_{1s})]}$$

The general probability P_j that a transition is made to stat j is given by

$$P_j = |a_j(\infty)|^2 = \frac{(e|\vec{E}_0|\tau)^2 \langle j|z|1s\rangle^2}{\hbar^2[1 + \tau^2(\omega_j - \omega_{1s})^2]}$$

This probability is dimensionless. It needs to be less than unity for this theory to be valid.

(a.) For the state $j = 2s$ the probability is zero. It vanishes because the matrix element of z is zero: $\langle 2s|z|1s\rangle = 0$ due to parity.

[II-8] Continued.

Both S-states have even parity and z has odd parity.

(b.) For the state $j = 2P$ the transition is allowed to the $L = 1$, $M = 0$ orbital state, which is called $2P_z$. The matrix element is similar what is found for the Stark effect. The $2P$ eigenstates for $L = 1$, $S = 0$ is

$$|10\rangle = \frac{z}{\sqrt{32\pi a_0^5}} e^{-r/2a_0}$$ and that

for the 1S state is $(-r/a_0)/\sqrt{\pi a_0^3}$.

The integral becomes

$$\langle 2P_z | z | 1S \rangle = \frac{2\pi}{\pi a_0^4 \sqrt{32}} \int_0^\infty dr\, r^4 e^{-3r/2a_0} \int_0^\pi d\theta \sin\theta \cos^2\theta$$

$$\langle 2P_z | z | 1S \rangle = \frac{1}{3\sqrt{2}\, a_0^4} \int_0^\infty dr\, r^4 e^{-3r/2a_0} = a_0 \left(\frac{2^{3/2}}{3} \right)^5,$$

and a_0 is the Bohr radius of the hydrogen atom.

(II-7.) Solution

(a.) Van der Waal's equation of state for a real gas is

$$\left(p+\frac{a}{V^2}\right)(V-b)=nRT$$

(b.) On the basis of the state equation for an ideal gas, we account for the intrinsic volume of a real gas molecules by introducing a constant b, and for the attractive forces among the molecules by introducing a pressure correction (a/V^2).

(c.) From $\left(p+\frac{a}{V^2}\right)(V-b)=nRT$ we have,

$$p=\frac{nRT}{V-b}-\frac{a}{V^2}$$

so that
$$\left(\frac{\partial p}{\partial V}\right)_T=\frac{-nRT}{(V-b)^2}+\frac{2a}{V^3}$$

$$\left(\frac{\partial^2 p}{\partial V^2}\right)_T=\frac{2nRT}{(V-b)^3}-\frac{6a}{V^4}$$

At the critical point, we have $\left(\frac{\partial p}{\partial V}\right)_T=0$ $\left(\frac{\partial^2 p}{\partial V^2}\right)_T=0$, so that

$$V_c=3b,\ p_c=\frac{a}{27b^2},\ nRT_c=\frac{8a}{27b}$$

namely: $a=3p_cV_c^2$, $b=V_c/3$.

(II-7) Note: using the form given in the problem, equivalently:

$$\left(P + a\left(\frac{N}{V}\right)^2\right)(V - bN) = NkT$$

$$P = \frac{NkT}{V - bN} - a\left(\frac{N}{V}\right)^2$$

at the critical point:

$$\left.\begin{aligned} \left(\frac{\partial P}{\partial V}\right)_{T,N} &= -\frac{NkT}{(V-bN)^2} + 2a\frac{N^2}{V^3} = 0 \\ \left(\frac{\partial^2 P}{\partial V^2}\right)_{T,N} &= \frac{2NkT}{(V-bN)^3} - 6a\frac{N^2}{V^4} = 0 \end{aligned}\right\} \Longrightarrow$$

$$\boxed{\begin{aligned} V_c &= 3Nb \\ kT_c &= \frac{8}{27}\frac{a}{b} \\ P_c &= \frac{1}{27}\frac{a}{b^2} \end{aligned}}$$

(both forms of solution are equivalent and accepted.)

II-8

II-8

$$F(T,V,N) = F^{ideal}(T,V,N) - \frac{2}{3} N e^2 \left(\frac{4\pi N e^2}{V k_B T}\right)^{1/2}$$

a) $$P = -\left(\frac{\partial F}{\partial V}\right)_{T,N} = \underbrace{-\left(\frac{\partial F^{ideal}}{\partial V}\right)_{T,N}}_{P(T,V)^{ideal}} - \frac{2}{3} N e^2 \left(+\frac{1}{2}\frac{1}{V}\right)\left(\frac{4\pi N e^2}{V k_B T}\right)^{1/2}$$

$$= \frac{N k_B T}{V} - \frac{1}{3}\frac{N e^2}{V}\left(\frac{4\pi N e^2}{V k_B T}\right)^{1/2}$$

b) $F = E - TS$

$E = F + TS$ and $S = -\left(\frac{\partial F}{\partial T}\right)_{V,N}$

$$S = -\left(\frac{\partial F}{\partial T}\right)_{V,N} = -\left(\frac{\partial F^{ideal}}{\partial T}\right)_{V,N} - \frac{2}{3} N e^2 \left(+\frac{1}{2}\frac{1}{T}\right)\left(\frac{4\pi N e^2}{V k_B T}\right)^{1/2}$$

$$= S^{ideal} - \frac{1}{3}\frac{N e^2}{T}\left(\frac{4\pi N e^2}{V k_B T}\right)^{1/2}$$

$$E = F + TS = F^{ideal} - \frac{2}{3} N e^2 \left(\frac{4\pi N e^2}{V k_B T}\right)^{1/2} + T S^{ideal} - \frac{1}{3} N e^2 \left(\frac{4\pi N e^2}{V k_B T}\right)^{1/2}$$

$$= \underbrace{F^{ideal} + T S^{ideal}}_{E^{ideal}} - N e^2 \left(\frac{4\pi N e^2}{V k_B T}\right)^{1/2} =$$

$$= E^{ideal} - N e^2 \left(\frac{4\pi N e^2}{V k_B T}\right)^{1/2} = \frac{3}{2} N k_B T - N e^2 \left(\frac{4\pi N e^2}{V k_B T}\right)^{1/2}$$

c) $$C_V = \left(\frac{\partial E}{\partial T}\right)_{V,N} = \frac{3}{2} N k_B + \frac{N e^2}{2T}\left(\frac{4\pi N e^2}{V k_B T}\right)^{1/2}$$

other possible way to obtain C_V:

$$C_V = \left(\frac{\delta Q}{\delta T}\right)_{V,N} = T\left(\frac{\partial S}{\partial T}\right)_{V,N} = -T\left(\frac{\partial^2 F}{\partial T^2}\right)_{V,N} =$$

$$= -T\left\{\left(\frac{\partial^2 F^{ideal}}{\partial T^2}\right)_{V,N} - \frac{2}{3}\cdot\frac{1}{2}\cdot\frac{3}{2}\frac{N e^2}{T^2}\left(\frac{4\pi N e^2}{V k_B T}\right)^{1/2}\right\} =$$

$$= C_V^{ideal} + \frac{N e^2}{2T}\left(\frac{4\pi N e^2}{V k_B T}\right)^{1/2} = \frac{3}{2} N k_B + \frac{N e^2}{2T}\left(\frac{4\pi N e^2}{V k_B T}\right)^{1/2}$$

II-9

Maxwell's speed distribution

$$n(v) = 4\pi N \left(\frac{m}{2\pi kT}\right)^{3/2} v^2 e^{-mv^2/2kT}$$

average speed $\quad v_{ave} = \frac{1}{N}\int_0^\infty v\, n(v)\, dv$

$$= 4\pi \left(\frac{m}{(2\pi kT)}\right)^{3/2} \int_0^\infty v^3 e^{-mv^2/2kT}$$

$$v_{av} = \sqrt{\frac{8kT}{\pi m}}$$

root mean square speed $\quad v_{rms} = \sqrt{\frac{3kT}{m}}$

$\boxed{\textbf{P22-7}}$ What is important here is the temperature; since the temperatures are the same then the average kinetic energies per particle are the same. Then

$$\frac{1}{2}m_1(v_{\mathrm{rms},1})^2 = \frac{1}{2}m_2(v_{\mathrm{rms},2})^2.$$

We are given in the problem that $v_{\mathrm{av},2} = 2v_{\mathrm{rms},1}$. According to Eqs. 22-18 and 22-20 we have

$$v_{\mathrm{rms}} = \sqrt{\frac{3RT}{M}} = \sqrt{\frac{3\pi}{8}}\sqrt{\frac{8RT}{\pi M}} = \sqrt{\frac{3\pi}{8}}v_{\mathrm{av}}.$$

Combining this with the kinetic energy expression above,

$$\frac{m_1}{m_2} = \left(\frac{v_{\mathrm{rms},2}}{v_{\mathrm{rms},1}}\right)^2 = \left(2\sqrt{\frac{3\pi}{8}}\right)^2 = 4.71.$$

II-10

II-10 [10]

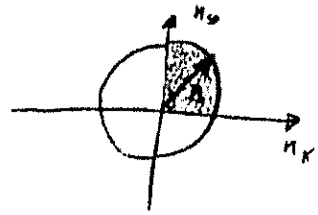
Consider a *two-dimensional photon gas* confined to an area A. What is the average number of photons in the system at temperature T?
Your answer must be expressed in terms of A, T, and of course, the necessary fundamental constants.

To keep your expressions relatively compact, the following relationship will come useful: $\int_0^\infty \frac{x^{\nu-1}}{e^x-1}dx = \Gamma(\nu)\zeta(\nu)$, where $\Gamma(\nu)$ is the gamma function and $\zeta(\nu)$ is the Rieman zeta function. In particular, $\zeta(2) = \frac{\pi^2}{6}$.

$\vec{k} = \frac{\pi}{L}(n_x, n_y)$ $\quad n_{x,y} = 1, 2, 3, \ldots$

$$\nu = \frac{\omega}{2\pi} = \frac{kc}{2\pi} = \frac{c}{2\pi}\sqrt{k_x^2 + k_y^2} = \frac{c}{2\pi}\left(\frac{\pi}{L}\right)\underbrace{\sqrt{n_x^2+n_y^2}}_{\equiv n} = \frac{c}{2L}\, n$$

$$n = \frac{2L}{c}\nu$$



number of modes with frequency less than ν: $\mathcal{N}(\nu)$

$$\mathcal{N}(\nu) = 2 \times \frac{1}{4}\pi n^2 = \frac{1}{2}\pi \frac{4L^2}{c^2}\nu^2 = \frac{2A\pi}{c^2}\nu^2$$

density of modes: $\boxed{g(\nu) = \frac{\partial \mathcal{N}}{\partial \nu} = \frac{4A\pi}{c^2}\nu}$

$$\langle n_\nu \rangle = \frac{1}{e^{\beta h\nu} - 1}$$

(Bosons, chemical potential = 0 quasi particles)

$$N_{ph} = \int_0^\infty d\nu\, g(\nu)\langle n_\nu\rangle = \int_0^\infty d\nu\, g(\nu)\frac{1}{e^{\frac{h\nu}{kT}}-1} = \frac{4A\pi}{c^2}\int_0^\infty \frac{d\nu\,\nu}{e^{\frac{h\nu}{kT}}-1} =$$

$$= \frac{4A\pi}{c^2}\left(\frac{kT}{h}\right)^2\int_0^\infty \frac{dx\,x}{e^x-1} = \frac{4\pi A}{c^2}\left(\frac{kT}{h}\right)^2 \overbrace{\Gamma(2)\zeta(2)}^{1\cdot\pi^2/6} =$$

$$= \boxed{\frac{2\pi^3 A k^2}{3c^2h^2}T^2} = \boxed{\text{const.}\, A \cdot T^2}$$