Physics PhD Qualifying Examination
Part I – Wednesday, August 24, 2011

Name:____
(please print)
Identification Number:________

STUDENT: Designate the problem numbers that you are handing in for grading in the appropriate left hand boxes below. Initial the right hand box.
PROCTOR: Check off the right hand boxes corresponding to the problems received from each student. Initial in the right hand box.

	1	
	2	
	3	
	4	
	5	
	6	
	7	
	8	
	9	
	10	

Student's initials

problems handed in:

Proctor's initials

INSTRUCTIONS FOR SUBMITTING ANSWER SHEETS

1. DO NOT PUT YOUR NAME ON ANY ANSWER SHEET. EXAMS WILL BE COLLATED AND GRADED BY THE ID NUMBER ABOVE.
2. Use at least one separate preprinted answer sheet for each problem. Write on only one side of each answer sheet.
3. Write your **identification number** listed above, in the appropriate box on each preprinted answer sheet.
4. Write the **problem number** in the appropriate box of each preprinted answer sheet. If you use more than one page for an answer, then number the answer sheets with both problem number and page (e.g. Problem 9 – Page 1 of 3).
5. Staple together all the pages pertaining to a given problem. Use a paper clip to group together all eight problems that you are handing in.
6. Hand in a total of *eight* problems. A passing distribution will normally include at least three passed problems from problems 1-5 (Mechanics) and three problems from problems 6-10 (Electricity and Magnetism). **DO NOT HAND IN MORE THAN EIGHT PROBLEMS.**
7. **YOU MUST SHOW ALL YOUR WORK.**

[I-1] [10]

Suppose the Moon were to have the same mass as the Earth, and you are trying to throw one of your physics books from the Earth to the Moon. With what minimum velocity must the book leave the surface of the Earth?

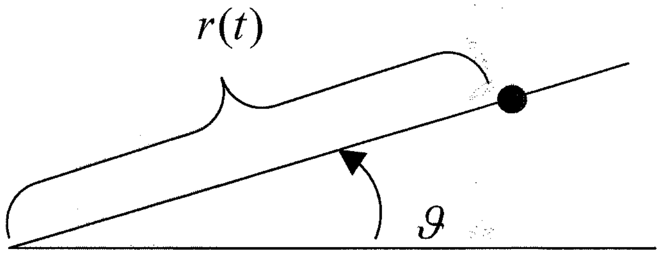
Neglect the relative motion of the Earth and the Moon, and the rotation of the Earth. The mass of the Earth is $M_E = 6.0 \times 10^{24}$ kg, the radius of the Earth is $R_E = 6.4 \times 10^6$ m, and the distance from the center of the Earth to the center of the Moon is R_{EM} 3.8×10^{8} m.

Compare your answer to the escape velocity from Earth alone. The gravitational constant is $G = 6.67 \times 10^{-11}$ $Nm^2{=}kg^2$.

[I-2] [10]

A particle of mass m *initially rests* on a smooth horizontal plane. The plane is then raised to an inclination angle ϑ at a constant rate, $\vartheta = \alpha t$, causing the particle to move down the plane. Determine the full motion of the particle, i.e., explicitly solve for $r(t)$.

$$r(0) = r_o$$



[I-3] **[10]**

Carbon dioxide, CO_2, has an equilibrium structure with three atoms aligned along an axis with the carbon atom located at the center of the molecule. The carbon atom is connected to the two oxygen atoms with chemical bonds. At finite temperatures, the relative positions of these atoms are subjected to thermal motion. The thermal motion can be considered as a superposition of normal modes of atomic motions. The chemical bonds can be considered as springs which follow Hooks law with a spring constant k. The displacements of the three atoms' from their equilibrium position are x_1 (O), x_2 (C), and x_3 (O). The mass of carbon atom is M and the mass of oxygen atom is m. Assume all atoms only can move along the long axis of the molecule. Write down the equations of motion. Find the eigenfrequencies and the eigenvectors for each of modes.

[I-4] **[10]**

A particle of mass m is moving under the influence of a central potential (with a fixed center),

$$U(r) = k\ln(r),$$

where $k > 0$ is a constant. The particle performs circular motion with a radius r_o. Determine the *frequency of small oscillations* ω_o about this circular orbit. Your answer must be expressed in terms of m, k, and r_o.

[I-5] **[4,6]**

One of the K_L meson decay modes is to three neutral pions:

$$K_L \rightarrow 3\pi^0$$

The masses are $m_K = 498$ MeV/c^2 and $m_\pi = 135$ MeV/c^2.

(a) What is the kinetic energy of pion number 1 if pion number 3 is at rest? Give your answer in MeV.

(b) What is the kinetic energy of pion number 1 if pion number 2 and pion number 3 go the same direction as each other with identical energies? Give your answer in MeV.

[I-6] **[10]**

(a) Write the general solution for the potential inside an empty conducting sphere of radius R (i.e., no charge distribution inside), with a defined potential $V(\theta',\phi')$ at the surface of the sphere.

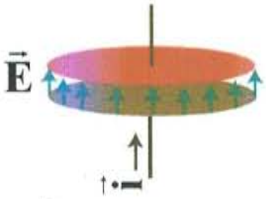
(b) Solve the special case where

$$V(\theta') = \begin{cases} +V_0 & \text{for } 0 < \theta' < \pi/2 \\ -V_0 & \text{for } \pi/2 < \theta' < \pi \end{cases}$$

along the z-axis (i.e., for $\theta = 0$).

[I-7] **[10]**

A circular parallel-plate capacitor with a radius R is being charged with current $\vec{\imath}$ as shown in the figure. Assume that the electric field is uniform between the plates.



(a) What is the magnitude of the induced magnetic field between the plates and at the position $r \leq R$ and $r > R$ from the center?

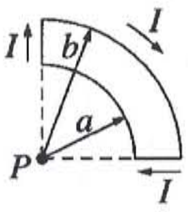
(b) What is the direction of the induced magnetic field when viewed from the top plate's side? (justify the reason).

(c) Between the plates, what is the magnitude and direction of the displacement current at the distance r=R/5 from the center in terms of $\vec{\imath}$?

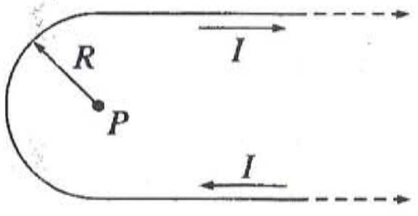
[I-8] **[10]**

(a) Write down the Biot-Savart law.

(b) Currents I are running in copper wires as shown below. Find the magnetic field at point P for each of the steady current configurations shown in the diagrams (1) and (2).



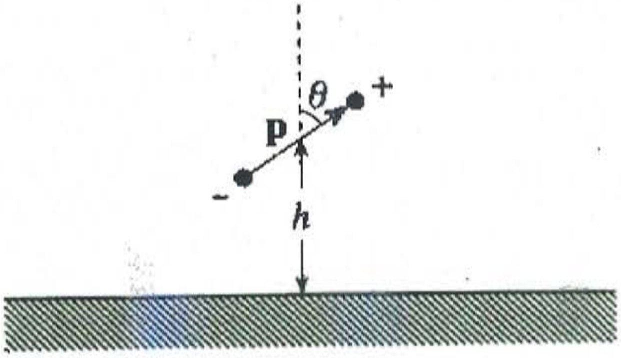
(1)



(2)

[I-9] **[4,6]**

An electric dipole of moment p is placed at a height h above a perfectly conducting plane and makes an angle θ with respect to the normal to the plane (see figure below).



Figure

(a) Indicate the position and orientation of the image dipole and the direction of the force felt by the dipole.

(b) Calculate the work required to remove the dipole to infinity.

[I-10] [10]

Consider two charges q_A and q_B. Charge q_A is at rest at the origin in system S.

(a) Charge q_B flies by at speed v on a trajectory parallel to the x-axis, but at $y=d$. What is the electromagnetic force on q_B as it crosses the y-axis?

(b) Now study the same problem from system S', q_B is at rest in S' and S' moves to the right with speed v. What is the force on q_B when q_A passes the y' axis?

Part I Solutions

Problem I-1

Suppose the Moon were to have the same mass as the Earth, and you are trying to throw one of your physics books from the Earth to the Moon. With what minimum velocity must the book leave the surface of the Earth?

Neglect the relative motion of the Earth and the Moon, and the Earths rotation. The mass of the Earth is, $M_E = 6.0 \times 10^{24} kg$, the radius of the Earth is, $R_E = 6.4 \times 10^6 m$, and the distance from the center of the Earth to the center of the Moon is $R_{EM} = 3.8 \times 10^8 m$.

Compare your answer to the escape velocity from Earth alone. The gravitational constant is $G = 6.67 \times 10^{-11} Nm^2/kg^2$.

Solution

We can determine the escape velocity using conservation of energy, where energy is given by the sum of potential energy due to gravity and kinetic energy. In the case of Earth along the potential is given by

$$\phi(r) = -G\frac{M_E m}{r}$$

where m is the mass of the book. The book will escape if its initial kinetic energy is high enough to overcome the potential at $r = R_E$. Thus

$$\frac{1}{2}mv_E^2 = -G\frac{M_E m}{R_E}$$

and

$$v_E = \sqrt{2G\frac{M_E}{R_E}} = 11km/s$$

In the Earth-Moon the potential is,

$$\phi(r) = -G\frac{M_E m}{r} - G\frac{M_M m}{|R_{EM} - r|}$$

Using $M_E = M_M$, as stated in the problem, the potential is a symmetric double well. In order for the particle to leave the surface of the earth, the kinetic energy must be high enough to overcome a saddle point right in the middle between the Earth and Moon.

The condition for escape velocity is

$$\frac{1}{2}mv_E^2 - GM_E m(\frac{1}{R_E} + \frac{1}{R_{EM} - R_E}) = -\frac{4GM_E m}{R_{ME}}$$

This gives $v_E = 7.7km/s$.

homogenous solutions: $r_H(t) = Ae^{\alpha t} + Be^{-\alpha t}$

particular solution: $r_p = C\sin(\alpha t)$,
→ **C** to be determined

$$-C\alpha^2\sin(\alpha t) = C\alpha^2\sin(\alpha t) - g\sin(\alpha t)$$

$$g = 2C\alpha^2 \Rightarrow \boxed{C = \frac{g}{2\alpha^2}}$$

Thus, the general solution:

$$r(t) = r_H(t) + r_p(t) = \boxed{Ae^{\alpha t} + Be^{-\alpha t} + \frac{g}{2\alpha^2}\sin(\alpha t)}$$

$\boxed{r(0) = r_0 \ , \ \dot{r}(0) = 0}$:

$$\begin{cases} A + B = r_0 \\ \alpha A - \alpha B + \frac{g}{2\alpha} = 0 \end{cases} \Rightarrow \begin{cases} A + B = r_0 \\ A - B = -\frac{g}{2\alpha^2} \end{cases}$$

$$A = \frac{1}{2}\left[r_0 - \frac{g}{2\alpha^2}\right] \ , \ B = \frac{1}{2}\left[r_0 + \frac{g}{2\alpha^2}\right]$$

$$\boxed{r(t) = r_0\cosh(\alpha t) - \frac{g}{2\alpha^2}\sinh(\alpha t) + \frac{g}{2\alpha^2}\sin(\alpha t)}$$

+

$$\begin{pmatrix} -\omega^2 m_1 + k & -k & 0 \\ -k & -\omega^2 m_2 + 2k & -k \\ 0 & -k & -\omega^2 m_1 + k \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} = 0$$

A condition of these equations to have non zero solution is the determinant of the matrix is zero.

$$\rightarrow \begin{vmatrix} -\omega^2 m_1 + k & -k & 0 \\ -k & -\omega^2 m_2 + 2k & -k \\ 0 & -k & -\omega^2 m_1 + k \end{vmatrix} = 0$$

$$\rightarrow \left(-\omega^2 m_1 + k\right)\left(-\omega^2 m_2 + 2k\right)\left(-\omega^2 m_1 + k\right) - 2k^2\left(-\omega^2 m_1 + k\right) = 0$$

$$\rightarrow \left(-\omega^2 m_1 + k\right)^2\left(-\omega^2 m_2 + 2k\right) - 2k^2\left(-\omega^2 m_1 + k\right) = 0$$

$$\rightarrow \left(-\omega^2 m_1 + k\right)\left[\left(-\omega^2 m_1 + k\right)\left(-\omega^2 m_2 + 2k\right) - 2k^2\right] = 0$$

$$\left(-\omega^2 m_1 + k\right) = 0 \rightarrow \omega = \sqrt{\frac{k}{m_1}}$$

$$\left[\left(-\omega^2 m_1 + k\right)\left(-\omega^2 m_2 + 2k\right) - 2k^2\right] = 0$$

$$\rightarrow m_1 m_2 \omega^4 - k\left(2m_1 + m_2\right)\omega^2 + 2k^2 - 2k^2 = 0$$

$$\rightarrow m_1 m_2 \omega^4 - k\left(2m_1 + m_2\right)\omega^2 = 0$$

$$\rightarrow \left[m_1 m_2 \omega^2 - k\left(2m_1 + m_2\right)\right]\omega^2 = 0$$

$\rightarrow$

There are three solutions.

$$\omega_1 = 0,$$

$$\omega_2 = \sqrt{\frac{k\left(2m_1 + m_2\right)}{m_1 m_2}}$$

$$\omega_3 = \sqrt{\frac{k}{m_1}}$$

Substitute $\omega = \sqrt{\frac{k}{m_1}}$

into the equation of motion.

$$\left(-\omega^2 m_1 + k\right)A_1 - kA_2 = 0 \rightarrow \left(-k + k\right)A_1 - kA_2 = 0 \rightarrow -kA_2 = 0 \rightarrow A_2 = 0$$

Similarly, from $-kA_2 + \left(-\omega^2 m_1 + k\right)A_3 = 0 \rightarrow -kA_2 = 0$

+

$$A_3 = -\frac{m_2}{2m_1} A_2$$

Eigen vector for

$\omega = \sqrt{\frac{k(2m_1+m_2)}{m_1 m_2}}$ is

$\mathbf{A} = A(-\varepsilon \quad 1 \quad -\varepsilon)$ where $\varepsilon = \frac{m_2}{2m_1} > 0$

The oxygen atoms and the carbon atom moves opposite direction. Asymmetric vibration.

For $\omega = 0$, substitute into

$$\left(-\omega^2 m_1 + k\right) A_1 - kA_2 = 0$$

→$(k)A_1 - kA_2 = 0$→$A_1 = A_2$

→Similarly from $-kA_2 + \left(-\omega^2 m_1 + k\right) A_3 = 0$.

$A_2 = A_3$

→ $A_1 = A_2 = A_3$

→ $\mathbf{A} = A(1 \quad 1 \quad 1)$

All atoms move same amount in same direction.

→ Translational motion

$$\ddot{r} = \frac{\ell^2}{m^2 r^3} - g(r) \qquad , \text{ where } g(r) \equiv \frac{k}{m}\frac{1}{r}$$

Must expand about $r = r_0$

$$r = r_0 + x \qquad (x << r_0)$$

$$\ddot{x} = \frac{\ell^2}{m^2 r_0^3} - \frac{3\ell^2}{m^2 r_0^4} x - g(r_0) - g'(r_0) x$$

$$\ddot{x} = \underbrace{\left[\frac{\ell^2}{m^2 r_0^3} - g(r_0)\right]}_{= 0 \text{ for circular orbit @ } r_0} - \left[\frac{3\ell^2}{m^2 r_0^4} + g'(r_0)\right] x$$

$$\ddot{x} = -\left[\frac{3\ell^2}{m^2 r_0^4} + g'(r_0)\right] x$$

Frequency of small oscillations: see next page

$$\omega_0^2 \equiv \frac{3\ell^2}{m^2 r_0^4} - \frac{k}{m}\frac{1}{r_0^2} = \frac{3(m r_0^2 k)}{m^2 r_0^4} - \frac{k}{m}\frac{1}{r_0^2} =$$

$$= 3\frac{k}{m}\frac{1}{r_0^2} - \frac{k}{m}\frac{1}{r_0^2} = \underline{\frac{2k}{m}\frac{1}{r_0^2}}$$

$$\omega_0 = \sqrt{\frac{2k}{m}\frac{1}{r_0^2}} = \boxed{\sqrt{\frac{2k}{m}}\,\frac{1}{r_0}} \quad ✓$$

Alternative Solution:
(I-4)

Alternatively, using the effective radial potential, $U_{eff}(r)$:

$$E = \frac{1}{2} m \dot{r}^2 + \frac{1}{2} m r^2 \dot{\theta}^2 + U(r)$$

$$= \frac{1}{2} m \dot{r}^2 + \frac{1}{2} m r^2 \left(\frac{\ell^2}{m^2 r^4} \right) + U(r)$$

$$= \frac{1}{2} m \dot{r}^2 + \frac{1}{2} \frac{\ell^2}{m r^2} + U(r) = \frac{1}{2} m \dot{r}^2 + U_{eff}(r)$$

where $\boxed{U_{eff}(r) = \frac{\ell^2}{2mr^2} + U(r)}$

effective (radial) equation of motion: $\left(\frac{\partial U}{\partial r} = m g(r) \right)$

$$m \ddot{r} = - \frac{\partial U_{eff}}{\partial r} = \frac{\ell^2}{m r^3} - \frac{\partial U}{\partial r}$$

$$m \ddot{r} = \frac{\ell^2}{m r^3} - m g(r)$$

$$\boxed{\ddot{r} = \frac{\ell^2}{m^2 r^3} - g(r)}$$

(same as obtained earlier)

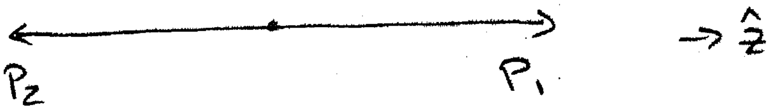
freq. of small oscillations (about stable circular orbit):

$$\boxed{m \omega^2 = \left. \frac{\partial^2 U_{eff}}{\partial r^2} \right|_{r_0}} = \frac{3 \ell^2}{m r_0^4} + m \left. \frac{dg}{dr} \right|_{r_0} =$$

$\boxed{\text{I-5}}$

$c=1.$

(a)



$$P_1 = (E, |\vec{p}|\hat{z})$$

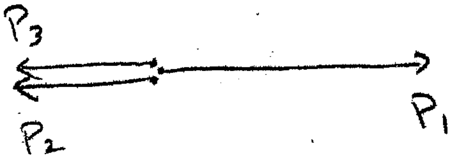
$$P_2 = (E, -|\vec{p}|\hat{z})$$

$$P_3 = (m_\pi, \vec{0})$$

$$m_K = 2E + m_\pi$$

$$E = \frac{1}{2}(m_K - m_\pi)$$

(b)



$$P_1 = (E, |\vec{p}|\hat{z})$$

$$P_2 = (E', -\tfrac{1}{2}|\vec{p}|\hat{z}) = P_3$$

$$E' = \sqrt{\frac{1}{4}|\vec{p}|^2 + m_\pi^2} = \sqrt{\frac{1}{4}(E^2 - m_\pi^2) + m_\pi^2}$$

$$m_K = E + 2E'$$

Problem I-6

i. Write the general solution for the potential inside an empty (i.e., no charge distribution inside) conducting sphere of radius R, with a defined potential distribution at the surface $\phi(R, \theta, \phi)$.

ii) Solve the special case where

$$V(\theta) = +V_0 \qquad \text{for} 0 \leq \theta \leq \pi/2 \tag{0.1}$$

$$= -V_0 \qquad \text{for} \pi/2 \leq \theta \leq \pi \tag{0.2}$$

along the z-axis (i.e., for $\theta = 0$).

Solution:

i. In spherical coordinates, the Green's function can be written as

$$G(\vec{x}, \vec{x}') = \frac{1}{(x^2 + x'^2 - 2xx'cos\gamma)^{1/2}} - \frac{1}{(x^2x'^2/R^2 + R^2 - 2xx'cos\gamma)^{1/2}}$$

and the Dirichlet boundary condition gives a surface charge density

$$\frac{\partial G}{\partial n'}|_{x'=R} = -\frac{(x^2 - R^2)}{a(x^2 + R^2 - 2\ R\ x\ cos\gamma)^{3/2}}$$

such that

$$\phi(\vec{x}) = \frac{1}{4\pi} \int d\Omega' \phi(R, \theta', \phi') \frac{R(x^2 - R^2)}{(x^2 + R^2 - 2\ R\ x\ cos\gamma)^{3/2}}$$

where

$$cos\gamma = cos\theta cos\theta' + sin\theta sin\theta' cos(\phi - \phi')$$

ii. For the specified potential,

$$\phi(x, \theta, \phi) = \frac{VR(x^2 - R^2)}{4\pi} \int_0^{2\pi} d\phi' \int_0^1 d(cos\theta') [\frac{1}{(R^2 + x^2 - 2Rxcos\gamma)^{3/2}} - \frac{1}{(R^2 + x^2 + 2Rxcos\gamma)^{3/2}}]$$

Considering the special case of $\theta = 0$, then $cos\gamma = cos(\theta')$ and

$$\phi(z) = V[1 - \frac{(z^2 - R^2)}{z\sqrt{z^2 + R^2}}]$$

Notice that at $z = R$,

$$\phi = V,$$

and at large distances goes asymptotically as

$$\phi \sim 3VR^2/2z^2.$$

+

For $r > R$,

$$\oint B ds \cos 0^o = \oint B ds = 2\pi r B = \mu_0 \varepsilon_0 \frac{d\Phi_E}{dt} = \mu_0 \varepsilon_0 \left(\pi R^2\right) \frac{dE}{dt}$$

$$\rightarrow 2\pi r B = \mu_0 \varepsilon_0 \left(\pi R^2\right) \frac{dE}{dt}$$

$$\rightarrow B = \mu_0 \varepsilon_0 \left(\frac{R^2}{2r}\right) \frac{dE}{dt}$$

Direction is counter clock wise.

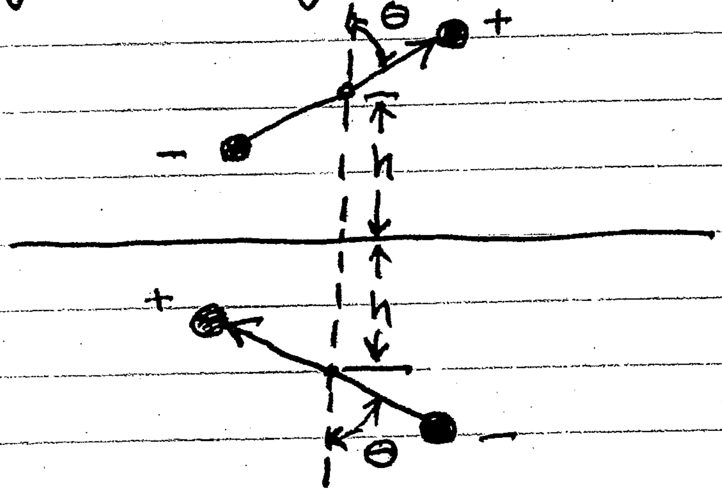
$$\oint \mathbf{B} \cdot d\mathbf{s} = \mu_0 \varepsilon_0 \frac{d\Phi_E}{dt} \equiv \mu_0 i_d = \mu_0 i_{d,Max} \frac{\pi r^2}{\pi R^2} = \mu_0 i \frac{(R/5)^2}{R^2} = \frac{\mu_0 i}{25}$$

$\rightarrow i_d = \frac{i}{25}$, direction is bottom to top plate.

Ph.D. Qu.E. Aug. 2011 (John Schroeder)

[I-9] Solution:

(i) The dipole is attracted to the plane, as seen from the position of the image charges (see figure below):



(ii) The field at a point $\vec{r}$ due to a dipole at the origin is given by

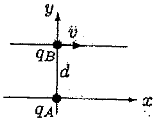
$$\vec{E}(\vec{r}) = \frac{3\vec{r}(\vec{r}\cdot\vec{p}) - \vec{p}r^2}{r^5}$$

The potential energy U of the dipole in the field of another dipole is given by $-\vec{p}\cdot\vec{E}$. Therefore, for two dipoles $\vec{p}_1$ and $\vec{p}_2$

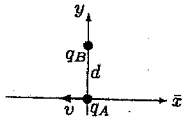
$$U = \frac{-3(\vec{r}\cdot\vec{p}_2)(\vec{r}\cdot\vec{p}_1) + (\vec{p}_1\cdot\vec{p}_2)r^2}{r^5}$$

where $\vec{r}$ is the vector from dipole 1 to dipole 2. Extra care must be taken here since this is an image problem and not one where a single dipole remains fixed and the

Problem 12.44



(a) Fields of A at B: $\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{q_A}{d^2}\hat{\mathbf{y}}$; $\mathbf{B} = 0$. So force on q_B is $\boxed{\mathbf{F} = \frac{1}{4\pi\epsilon_0}\frac{q_A q_B}{d^2}\hat{\mathbf{y}}.}$



(b) (i) From Eq. 12.68: $\boxed{\bar{\mathbf{F}} = \frac{\gamma}{4\pi\epsilon_0}\frac{q_A q_B}{d^2}\hat{\mathbf{y}}.}$ (*Note:* here the particle is at rest in $\bar{S}$.)

(ii) From Eq. 12.92, with $\theta = 90°$: $\bar{E} = \frac{1}{4\pi\epsilon_0}\frac{q_A(1-v^2/c^2)}{(1-v^2/c^2)^{3/2}}\frac{1}{d^2}\hat{\mathbf{y}} = \frac{\gamma}{4\pi\epsilon_0}\frac{q_A}{d^2}\hat{\mathbf{y}}$ (this also follows from Eq. 12.108).

$\bar{\mathbf{B}} \neq 0$, but since $v_B = 0$ in $\bar{S}$, there is no magnetic force anyway, and $\boxed{\bar{\mathbf{F}} = \frac{\gamma}{4\pi\epsilon_0}\frac{q_A q_B}{d^2}\hat{\mathbf{y}}}$ (as before).

Problem 12.45

For the solution also refer to:

David J. Griffith, "Introduction to Electrodynamics" 3rd edition, Prentice-Hall 1999
Chapters 12.2.4 and 12.3.2.

Chapter 12.2.4

be retained.

Because $\mathbf{F}$ is the derivative of momentum with respect to *ordinary* time, it shares the ugly behavior of (ordinary) velocity, when you go from one inertial system to another: both the numerator *and the denominator* must be transformed. Thus,[12]

$$\bar{F}_y = \frac{d\bar{p}_y}{d\bar{t}} = \frac{dp_y}{\gamma\, dt - \dfrac{\gamma\beta}{c}\, dx} = \frac{dp_y/dt}{\gamma\left(1 - \dfrac{\beta}{c}\dfrac{dx}{dt}\right)} = \frac{F_y}{\gamma(1-\beta u_x/c)}, \tag{12.66}$$

and similarly for the z component:

$$\bar{F}_z = \frac{F_z}{\gamma(1-\beta u_x/c)}.$$

The x component is even worse:

$$\bar{F}_x = \frac{d\bar{p}_x}{d\bar{t}} = \frac{\gamma\, dp_x - \gamma\beta\, dp^0}{\gamma\, dt - \dfrac{\gamma\beta}{c}\, dx} = \frac{\dfrac{dp_x}{dt} - \beta\dfrac{dp^0}{dt}}{1 - \dfrac{\beta}{c}\dfrac{dx}{dt}} = \frac{F_x - \dfrac{\beta}{c}\left(\dfrac{dE}{dt}\right)}{1 - \beta u_x/c}.$$

We calculated dE/dt in Eq. 12.64; putting that in,

$$\bar{F}_x = \frac{F_x - \beta(\mathbf{u}\cdot\mathbf{F})/c}{1 - \beta u_x/c}. \tag{12.67}$$

Only in one special case are these equations reasonably tractable: *If the particle is (instantaneously) at rest in S*, so that $\mathbf{u} = 0$, then

$$\bar{\mathbf{F}}_\perp = \frac{1}{\gamma}\mathbf{F}_\perp, \quad \bar{F}_\parallel = F_\parallel. \tag{12.68}$$

Chapter 12.3.2

transformation rules:

$$\boxed{\begin{array}{lll} \bar{E}_x = E_x, & \bar{E}_y = \gamma(E_y - vB_z), & \bar{E}_z = \gamma(E_z + vB_y), \\ \bar{B}_x = B_x, & \bar{B}_y = \gamma\left(B_y + \dfrac{v}{c^2}E_z\right), & \bar{B}_z = \gamma\left(B_z - \dfrac{v}{c^2}E_y\right). \end{array}} \tag{12.108}$$

Physics PhD Qualifying Examination
Part II – Friday, August 26 2011

Name:____
(please print)
Identification Number:________

STUDENT: insert a check mark in the left boxes to designate the problem numbers that you are handing in for grading.
PROCTOR: check off the right hand boxes corresponding to the problems received from each student. Initial in the right hand box.

	1	
	2	
	3	
	4	
	5	
	6	
	7	
	8	
	9	
	10	

Student's initials

\# problems handed in:

Proctor's initials

INSTRUCTIONS FOR SUBMITTING ANSWER SHEETS

1. DO NOT PUT YOUR NAME ON ANY ANSWER SHEET. EXAMS WILL BE COLLATED AND GRADED BY THE ID NUMBER ABOVE.
2. Use at least one separate preprinted answer sheet for each problem. Write on only one side of each answer sheet.
3. Write your **identification number** listed above, in the appropriate box on the preprinted sheets.
4. Write the **problem number** in the appropriate box of each preprinted answer sheet. If you use more than one page for an answer, then number the answer sheets with both problem number and page (e.g. Problem 9 – Page 1 of 3).
5. Staple together all the pages pertaining to a given problem. Use a paper clip to group together all eight problems that you are handing in.
6. Hand in a total of *eight* problems. A passing distribution will normally include at least four passed problems from problems 1-6 (Quantum Physics) and two problems from problems 7-10 (Thermodynamics and Statistical Mechanics). **DO NOT HAND IN MORE THAN EIGHT PROBLEMS.**
7. **YOU MUST SHOW ALL YOUR WORK.**

[II-1] **[8,2]**

Consider the finite asymmetric potential well shown in the figure below for the discrete spectrum, $0 < E < V_2$.

(a) Obtain an equation for the discrete energy levels (you do not have to solve it). Make a sketch to graphically illustrate the solutions of this equation.

(b) Consider and discuss the special (symmetric) case where $V_1 = V_2$.

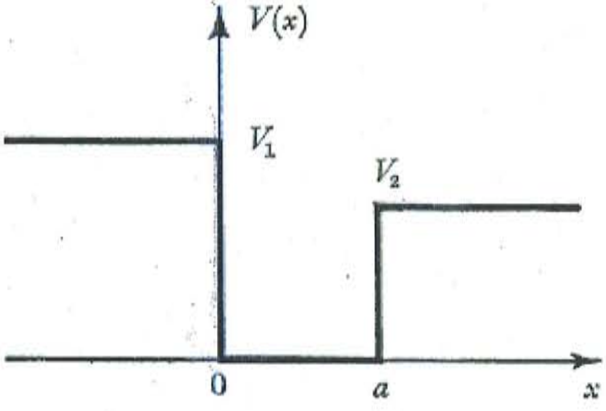


Fig. 1.

[II-2] **[10]**

Calculate the lowest-order correction to the energy of a one-dimensional simple quantum harmonic oscillator

$$H = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 x^2$$

perturbed by a potential

$$H_1 = \frac{1}{4}\alpha x^4 .$$

The ground-state wave function of the oscillator is given by $\psi_o(x) = \left(\frac{m\omega}{\hbar\pi}\right)^{1/4} \exp\left(-\frac{m\omega x^2}{2\hbar}\right)$.

[II-3] **[10]**

Show that for a system consisting of two identical particles with spin I, the ratio of the number of states symmetrical in the two spins to the number of states anti-symmetric in the two spins is equal to $(I+1)/I$.

[II-4] **[10]**

Using the Born approximation, evaluate the differential scattering cross section for scattering of particles of mass m and incident energy E by the repulsive spherical well with potential:

$$\begin{array}{ll} V(r)=V_0 & \text{for } 0<r\leq a \\ V(r)=0 & \text{for } r>a \end{array} .$$

[II-5] **[1,3,5,1]**

Here we consider one-dimensional quantum mechanics with a Gaussian wavefunction

$$\psi = N \exp(-\alpha x^2),$$

where N is a normalization constant.

(a) Find the normalization constant N such that the wavefunction has unit normalization.
(b) Find the uncertainty Δx.
(c) Find the uncertainty Δp.
(d) Compute $\Delta x\, \Delta p$ and compare to the uncertainty principle.

[II-6] [10]

An electronic two level system was in its ground state $|g\rangle$ for $t < 0$. An oscillating electric field is applied at $t = 0$ (and thereafter). The interaction between the system and the electric field is given by the Hamiltonian

$$H_1 = -\hat{\mu}E_o e^{i\omega t} - \hat{\mu}E_o e^{-i\omega t} ,$$

where $\hat{\mu}$ is the operator for the electric dipole moment in the direction of the electric field, and E_o is the amplitude of the electric field. The eigenvectors and eigenvalues of the unperturbed Hamiltonian are $|g\rangle$, $\varepsilon_g = \hbar\omega_g$ and $|e\rangle$, $\varepsilon_e = \hbar\omega_e$ for the ground state and for the excited state, respectively.

What is the probability of finding the system in its excited state $|e\rangle$ at time $= T$? Draw a sketch of the probability as a function of $\omega - \omega_{eg}$, where $\omega_{eg} = \omega_e - \omega_g$. Assume that the interaction between the electric field and the two-level system is small enough that you can treat it as a perturbation.

[II-7] **[10]**

A monoatomic gas obeys the van der Waals equation

$$P = \frac{NkT}{V - bN} - a\frac{N^2}{V^2},$$

and has the heat capacity $C_V = 3Nk/2$ in the limit of $V \to \infty$. (P is the pressure, V is the volume, N is the number of particles, k is the Boltzmann constant, and a, b are material-specific constants.)

a) Prove using thermodynamic identities and the equation of state, that

$$\left(\frac{\partial C_V}{\partial V}\right)_T = 0 .$$

b) Use the preceding result to determine the entropy of the van der Waals gas, $S(T,V)$, to within an additive constant.

c) Calculate the internal energy $U(T,V)$ to within an additive constant.

d) What is the final temperature when the gas is adiabatically and reversibly compressed from (V_1, T) to final volume V_2 ?

e) How much work is done in this compression?

[II-8] **[10]**

Calculate the difference between c_p and c_v for the ideal gas. Here c_p and c_v are the specific heats at constant pressure and constant volume, respectively. The ideal gas equation is given by:

$$PV = nRT$$

with thermodynamic variables pressure P, temperature T and volume V. R is the universal gas constant and n is the amount of mols. About the ideal gas, all you can use is the equation state given above. All other properties must be derived from there and from relevant fundamental thermodynamic identities. You must derive your answer from scratch, and show all your work, as always (i.e., this is not a memory test).

[II-9] [10]

The single-particle energy levels of a system of N *distinguishable* particles are $\varepsilon_1 = 0$ and $\varepsilon_2 = \varepsilon$. The degeneracies associated with the two energy levels are $g_1 = g_2 = 2$.

Obtain the entropy of the system $S(N,T)$. What is the limit of the entropy as $T \rightarrow 0$?

[II-10] [8,2]

A Fermi gas of spin ½ fermions of mass m is contained in a cubical box with side length L. The number of particles in the box is N. The ground state energy is U_0. The average kinetic energy of the particles in the ground state is U_0/N.

(a) Compute the average kinetic energy of the particles in terms of the Fermi energy ε_F.

(b) How does U_0 change if L is decreased while N is held constant?

QE. Ph.D Aug. 2011 (1.)

Part II
Solutions

[II-1] Solution

(a) $$\frac{\hbar^2}{2\mu}\frac{d^2\psi}{dx^2}+(E-V_1)\psi=0 \quad (x<0)$$

$$\frac{\hbar^2}{2\mu}\frac{d^2\psi}{dx^2}+E\psi = 0 \quad (0<x<a)$$

$$\frac{\hbar^2}{2\mu}\frac{d^2\psi}{dx^2}+(E-V_2)\psi=0 \quad (x>a)$$

let us use the following notation

$$\kappa_1 = \frac{\sqrt{2\mu(V_1-E)}}{\hbar},\ \kappa = \frac{\sqrt{(-2\mu E)}}{\hbar};\ \kappa_2 = \frac{\sqrt{2\mu(E-V_2)}}{\hbar}$$

hence the general solution in each of the three regions has the following form:

$$\psi = A_1 e^{(-\kappa_1 x)} + B_1 e^{(\kappa_1 x)} \quad (x<0)$$

$$\psi = A e^{(-\kappa x)} + B e^{(\kappa x)} \quad (0<x<a)$$

$$\psi = A_2 e^{(-\kappa_2 x)} + B_2 e^{(\kappa_2 x)} \quad (x>a)$$

Let us consider the discrete spectrum $E<V_2$; κ_1 and κ_2 are then real. If we put in the interval $0<x<a$, $\kappa = ik$, where k is real consequently,

$$\psi = \sin(kx+\delta) \quad (0<x<a)$$

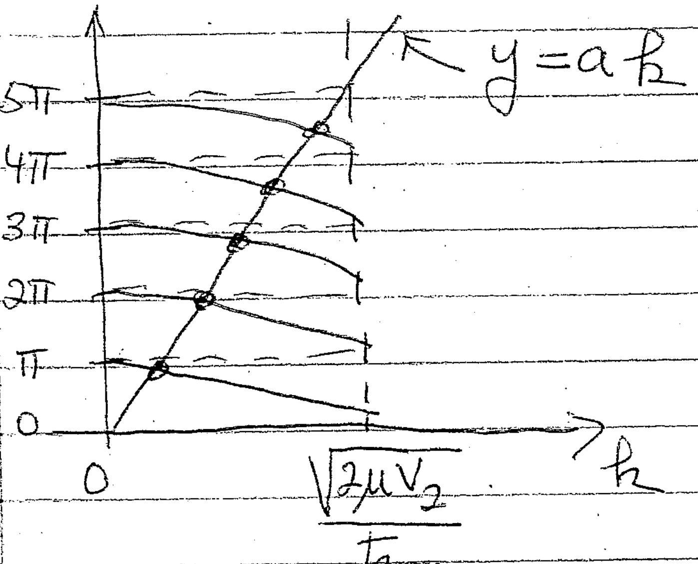
[II-1] Solution-continued.

a transcedental equation to determine the discrete energy levels

$$ka = n\pi - \sin^{-1}\frac{\hbar k}{\sqrt{2\mu V_1}} - \sin^{-1}\frac{\hbar k}{\sqrt{2\mu V_2}}$$

$$k = \frac{\sqrt{2\mu E}}{\hbar} > 0$$

values of k satisfying the above equation must be obtained graphically.



(b) Now for the symmetrical potential well $V_1 = V_2 = V$. It is seen that in this case there is always ~~one~~ at least one level for whatever the values of V and a are. If $\beta = \left[\frac{\sqrt{2\mu V}}{\hbar}\right] a \ll 1$, one may find without difficulty the value of the only discrete energy level.

~~Solution.~~

II-~~1~~.) 2

~~energy~~

$$E_1 = \langle \psi_0 | H_1 | \psi_0 \rangle = \int_{-\infty}^{+\infty} \psi_0^* H_1 \psi_0 \, dx$$

$$= \frac{1}{4} \alpha \left(\frac{m\omega}{\hbar\pi}\right)^{1/2} \int_{-\infty}^{+\infty} x^4 \exp\left(-\frac{m\omega x^2}{\hbar}\right)$$

$$= \frac{1}{4} \alpha \left(\frac{m\omega}{\hbar\pi}\right)^{1/2} \cdot 2 \int_0^{\infty} x^4 \exp\left(-\frac{m\omega x^2}{\hbar}\right) dx$$

$$= \frac{1}{4} \alpha \left(\frac{m\omega}{\hbar\pi}\right)^{1/2} \cdot 2 \; \frac{3\sqrt{\pi}}{8 \cdot \left(\frac{m\omega}{\hbar}\right)^{5/2}}$$

$$= \frac{1}{16} \alpha \cdot 3 \cdot \frac{\hbar^2}{m^2\omega^2}$$

$$= \frac{3}{16} \alpha \frac{\hbar^2}{m^2\omega^2}$$

$$\int_0^{\infty} x^n e^{-ax^2} dx = \frac{3\sqrt{\pi}}{2^3 a^{5/2}}$$

$n = 4 = 2k$

$k = 2$

Modern Physics
II-4 Scattering (Born Approx.) [10]

Using the Born approximation. Evaluate the differential scattering cross section for scattering of particles of mass m and incident energy E by the repulsive spherical well with potential

$V(r)=V_0$ for $0<r\leq a$
$V(r)=0$ for $r>a$.

II-4

~~4~~10

$$\frac{d\sigma}{d\Omega} = |F(\theta)|^2$$

$$F(\theta) = -\frac{2m}{\hbar^2 K}\int_0^{\infty} r\,V(r)\sin Kr\,dr$$

$$= -\frac{2m}{\hbar^2 K} V_0 \int_0^{a} r \sin Kr\,dr$$

$$= -\frac{2mV_0}{\hbar^2 K}\left[\frac{\sin Kr}{K^2} - \frac{r\cos Kr}{K}\right]_0^a$$

$$= -\frac{2mV_0}{\hbar^2 K}\left[\frac{\sin Ka}{K^2} - \frac{a\cos Ka}{K}\right]$$

$$= \hbar^2\left(\alpha - \alpha^2 \frac{1}{2\alpha}\right) = \hbar^2 \frac{\alpha}{2}$$

$$\Delta p = \hbar\sqrt{\frac{\alpha}{2}}$$

$$\Delta x \, \Delta p = \frac{\hbar}{2}$$

minimum uncertainty allowed by uncertainty principle.

+

→

$$i\hbar \sum_{n=g,e} \frac{\partial a_n(t)}{\partial t} | n\rangle e^{-i\omega_n t} = \sum_{n=g,e} a_n(t) H_1 | n\rangle e^{-i\omega_n t}$$

Multiplying $< e |$

$$i\hbar \frac{\partial a_e(t)}{\partial t} e^{-i\omega_e t} = \sum_{n=g,e} a_n(t) \langle e | H_1 | n\rangle e^{-i\omega_n t}$$

$$i\hbar \frac{\partial a_e(t)}{\partial t} = \sum_{n=g,e} a_n(t) \langle e | H_1 | n\rangle e^{-i(\omega_n - \omega_e)t}$$

Perturbation approximation is introduced by expanding the coefficient as follow.

$$a_k(t) = a_k^{(0)} + a_k^{(1)} + a_k^{(2)} + \cdots$$

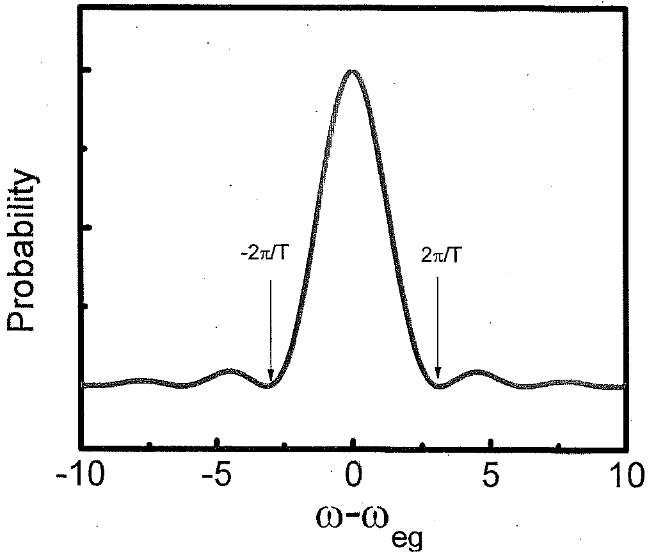
$$i\hbar \frac{\partial a_e^{(0)}(t)}{\partial t} + i\hbar \frac{\partial a_e^{(1)}(t)}{\partial t} = \sum_{n=g,e} a_n^{(0)}(t) \langle e | H_1 | n\rangle e^{-i(\omega_n - \omega_e)t} + \sum_{n=g,e} a_n^{(1)}(t) \langle e | H_1 | n\rangle e^{-i(\omega_n - \omega_e)t}$$

The system was initially in ground state,

$$i\hbar \frac{\partial a_e(t)}{\partial t} = \langle e | H_1 | g\rangle e^{-i\omega_{ge} t}$$

$$i\hbar \frac{\partial a_e(t)}{\partial t} = \langle e | -\hat{\mu} E_0 | g\rangle e^{-i\omega_{ge} t} e^{i\omega t} + \langle e | -\hat{\mu} E_0^{\dagger} | g\rangle e^{-i\omega_{ge} t} e^{-i\omega t}$$

Integrating the equation from t=0 to t=T.


Probability
-2π/T
2π/T
-10
-5
0
5
10
ω-ω_eg

[II-7] continued.

$$S(\tau, V) = \frac{3}{2} N \ln \tau + N \ln (V - Nb) + \text{const}$$

$$S(\tau, V) = N \ln \left[(V - Nb) \tau^{3/2} \right]$$

(c) The internal energy may be calculated from

$$\mathcal{E}(\tau, V) = \int \left.\frac{\partial \mathcal{E}}{\partial \tau}\right|_V d\tau + \int \left.\frac{\partial \mathcal{E}}{\partial V}\right|_\tau dV + \text{const.}$$

but $d\mathcal{E} = \tau\, dS - P\, dV$ hence,

$$\left.\frac{\partial \mathcal{E}}{\partial V}\right|_\tau = \tau \left.\frac{\partial S}{\partial V}\right|_\tau - P$$

and using the van der Waal equation and the expression for the entropy $S(\tau, V)$ we find that

$$\left.\frac{\partial \mathcal{E}}{\partial V}\right|_\tau = \frac{N\tau}{V - Nb} - \frac{N\tau}{V - Nb} + \frac{Na^2}{V^2} = \frac{N^2 a}{V^2}$$

thus, $\mathcal{E}(\tau, V)$ becomes

$$\mathcal{E}(\tau, V) = \int \frac{3}{2} N\, d\tau + \int \frac{N^2 a}{V^2} dV + \text{const}$$

$$\mathcal{E}(\tau, V) = \frac{3}{2} N\tau - \frac{N^2 a}{V} + \text{const}$$

~~$\mathcal{E}(\tau, V) = \frac{3}{2} N\tau$~~

$\boxed{\text{II - 8}}$

$H = E + PV$ $\qquad\qquad C_v = \left(\frac{\partial E}{\partial T}\right)_v \,,\quad C_p = \left(\frac{\partial H}{\partial T}\right)_p$

1) $$C_p = \left(\frac{\partial H}{\partial T}\right)_p = \left(\frac{\partial E}{\partial T}\right)_p + P\left(\frac{\partial V}{\partial T}\right)_p$$

2) $$dE = \left(\frac{\partial E}{\partial T}\right)_V dT + \left(\frac{\partial E}{\partial V}\right)_T dV$$

$$\left(\frac{\partial E}{\partial T}\right)_p = \left(\frac{\partial E}{\partial T}\right)_V + \left(\frac{\partial E}{\partial V}\right)_T\left(\frac{\partial V}{\partial T}\right)_p = C_v + \left(\frac{\partial E}{\partial V}\right)_T\left(\frac{\partial V}{\partial T}\right)_p$$

$$C_p = C_v + \left(\frac{\partial E}{\partial V}\right)_T\left(\frac{\partial V}{\partial T}\right)_p + P\left(\frac{\partial V}{\partial T}\right)_p$$

$$\boxed{C_p - C_v = \left(\frac{\partial V}{\partial T}\right)_p \left[\left(\frac{\partial E}{\partial V}\right)_T + P\right]}$$

3) $$dE = TdS - PdV$$

$$\left(\frac{\partial E}{\partial V}\right)_T = T\left(\frac{\partial S}{\partial V}\right)_T - P$$

Maxwell rel: $\left(\frac{\partial S}{\partial V}\right)_T = \left(\frac{\partial P}{\partial T}\right)_V$

$$\left(\frac{\partial E}{\partial V}\right)_T = T\left(\frac{\partial P}{\partial T}\right)_V - P$$

(important identity)

-1- $$\boxed{\left(\frac{\partial E}{\partial V}\right)_T + P = T\left(\frac{\partial P}{\partial T}\right)_V}$$

II-9] $Z = \sum_j g_j e^{-\beta \varepsilon_j}$ $\qquad \beta = \frac{1}{kT}$

$$Z = 2e^{0} + 2e^{-\beta\varepsilon} = 2(1+e^{-\beta\varepsilon})$$

$$\ln Z = \ln(2) + \ln(1+e^{-\beta\varepsilon})$$

$F(T,N) = -NkT \ln Z =$ (Helmholtz free energy)

$$= -NkT\ln(2) - NkT\ln(1+e^{-\varepsilon/kT})$$

$$S = -\left(\frac{\partial F}{\partial T}\right)_N = Nk\ln(2) + \frac{\partial}{\partial T}\left\{NkT\ln(1+e^{-\varepsilon/kT})\right\}$$

$$= Nk\ln(2) + Nk\ln(1+e^{-\varepsilon/kT}) + NkT\,\frac{e^{-\varepsilon/kT}\,\frac{\varepsilon}{kT^2}}{1+e^{-\varepsilon/kT}}$$

$$\boxed{= Nk\ln(2) + Nk\ln(1+e^{-\varepsilon/kT}) + Nk\left(\frac{\varepsilon}{kT}\right)\frac{1}{1+e^{\varepsilon/kT}}}$$

$\varepsilon/kT \gg 1$:] $S \simeq Nk\ln(2) + Nk\,e^{-\varepsilon/kT} + Nk\left(\frac{\varepsilon}{kT}\right)e^{-\varepsilon/kT}$

$$\simeq Nk\ln(2) + Nk\left(\frac{\varepsilon}{kT}\right)e^{-\varepsilon/kT} \to \boxed{Nk\ln(2)}$$

II-10

$$\epsilon_n = \frac{\hbar^2}{2m}\left(\frac{\pi n}{L}\right)^2$$

$$n = n_x^2 + n_y^2 + n_z^2$$

$$U_0 = (2)\left(\frac{1}{8}\right) 4\pi \int_0^{n_F} dn\, n^2 \epsilon_n$$

$$= \frac{\pi^3}{10m}\left(\frac{\hbar}{L}\right)^2 n_F^5$$

$$N = (2)\left(\frac{1}{8}\right) \frac{4\pi}{3} n_F^3 \Rightarrow n_F = \left(\frac{3N}{\pi}\right)^{1/3}$$

$$\epsilon_F = \frac{\hbar^2}{2m}\left(\frac{\pi n_F}{L}\right)^2$$

$$U_0 = \frac{3\hbar^2}{10m}\left(\frac{\pi n_F}{L}\right)^2 N = \frac{3}{5} N \epsilon_F$$

$$\therefore \quad \frac{U_0}{N} = \frac{3}{5}\epsilon_F$$

U_0 increases if L is decreased while holding N constant. Fermi repulsion.