Physics PhD Qualifying Examination
Part I – Wednesday, August 25, 2010

Name: ______________________________
(please print)
Identification Number: ________

STUDENT: Designate the problem numbers that you are handing in for grading in the appropriate left hand boxes below. Initial the right hand box.
PROCTOR: Check off the right hand boxes corresponding to the problems received from each student. Initial in the right hand box.

	1	
	2	
	3	
	4	
	5	
	6	
	7	
	8	
	9	
	10	

Student's initials
problems handed in:
Proctor's initials

INSTRUCTIONS FOR SUBMITTING ANSWER SHEETS

1. DO NOT PUT YOUR NAME ON ANY ANSWER SHEET. EXAMS WILL BE COLLATED AND GRADED BY THE ID NUMBER ABOVE.
2. Use at least one separate preprinted answer sheet for each problem. Write on only one side of each answer sheet.
3. Write your **identification number** listed above, in the appropriate box on each preprinted answer sheet.
4. Write the **problem number** in the appropriate box of each preprinted answer sheet. If you use more than one page for an answer, then number the answer sheets with both problem number and page (e.g. Problem 9 – Page 1 of 3).
5. Staple together all the pages pertaining to a given problem. Use a paper clip to group together all eight problems that you are handing in.
6. Hand in a total of *eight* problems. A passing distribution will normally include at least three passed problems from problems 1-5 (Mechanics) and three problems from problems 6-10 (Electricity and Magnetism). **DO NOT HAND IN MORE THAN EIGHT PROBLEMS.**
7. **YOU MUST SHOW ALL YOUR WORK.**

[I-1]   [10]

A uniform rope of mass M and length L is hanging from a ceiling. The gravitational acceleration is g.

How long does it take for a transverse wave to traverse the full length of the rope?

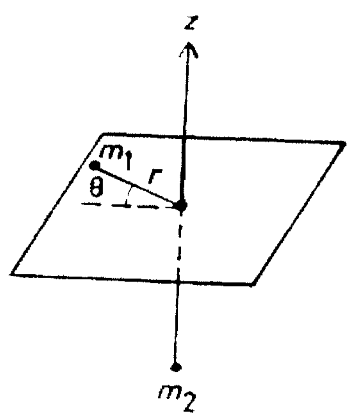
(You must *derive* your answer. Guessing or hand-waving the answer, or pulling it out of your memory, will yield zero credit.)

[I-2]   [3,7]

Two mass points m_1 and m_2 ($m_1 \neq m_2$) are connected by a massless string of length l passing through a hole in a horizontal table. The string and mass points move without friction with m_1 on the table and m_2 free to move in a vertical line.

(a) What initial velocity must m_1 be given so that m_2 will remain motionless a distance d below the surface of the table?

(b) If m_2 is slightly displaced in a vertical direction, small oscillations ensue. Use Lagrange's equations to find the period of these oscillations.



[I-3] **[10]**

A mass m moving in space is subject to a force whose potential energy is

$$V = V_0 \exp[(5x^2 + 5y^2 + 8z^2 - 8yz - 26ay - 8az)/a^2] ,$$

where the constants V_0 and a are positive. Show that V has one minimum point. Find the normal frequencies of vibration about the minimum.

[I-4] **[10]**

Determine the principle axis of inertia and principle moments of inertia of a uniformly solid hemisphere of radius b and mass m about its center of mass.

Hint: Use spherical coordinates

$$\begin{aligned} x &= r\cos(\phi)\sin(\theta) \\ y &= r\sin(\phi)\sin(\theta) \\ z &= r\cos(\theta) \end{aligned} .$$

[I-5] **[3,3,4]**

The electron has a rest mass of $9.1\text{x}10^{-31}$kg. On answering the following questions, relativistic effects must be considered.

(a) What is the rest energy of electron in units of eV?
(b) If the total energy of an electron is 2 times its rest energy, with what speed is the electron moving relative to the observer?
(c) Calculate the kinetic energy in units of eV.

[I-6] **[6,4]**

(a) Calculate the electrostatic potential within a conducting rectangular box for which all sides but one are grounded and the remaining side is at a potential V (at $z = 0$). Let the lengths of the box in the x-, y- and z-directions be a, b and c, respectively.

(b) Using the solution above, calculate the electrostatic potential within a conducting box for which all sides but two are grounded, and the two remaining sides are on opposite faces perpendicular to the z-direction and with potentials V_1 (at $z = 0$) and V_2 (at $z = c$). Let the lengths of the box in the x-, y- and z-directions be a, b and c, respectively.

[I-7] **[1,1,8]**

(a) Write down all four Maxwell's equations for a system with charge density, ρ and current density, $\mathbf{j}$.

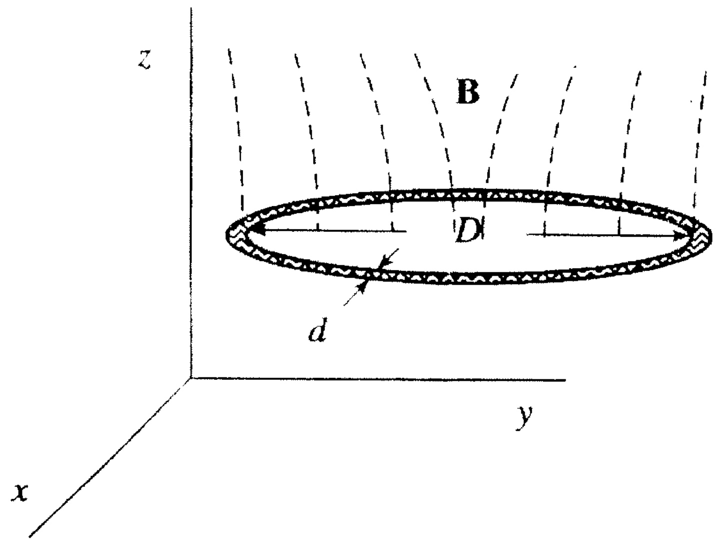
(b) Which equation represents Maxwell-Ampere's law ?

(c) Show that Maxwell's equations do not satisfy the charge continuity equation, $\frac{\partial \rho}{\partial t} + div\mathbf{j} = 0$, if the displacement current is not considered.

[1-8] **[10]**

A loop is in a magnetic field. A conducting circular loop made of wire of diameter d, resistivity ρ, and mass density ρ_m is falling from great height h in a magnetic field with a component $B_z = B_0(1+\kappa z)$, where κ is some constant. The loop of diameter D is always parallel to the x-y plane. Disregarding air resistance, find the terminal velocity of the loop.

Hint: From energy conservation, the work done by gravity during this stationary motion goes into Joule heating of the loop.



[1-9]   [2,4,1,2,1]

The following three equations each describe the electric field $\boldsymbol{E}$ of an electromagnetic wave:

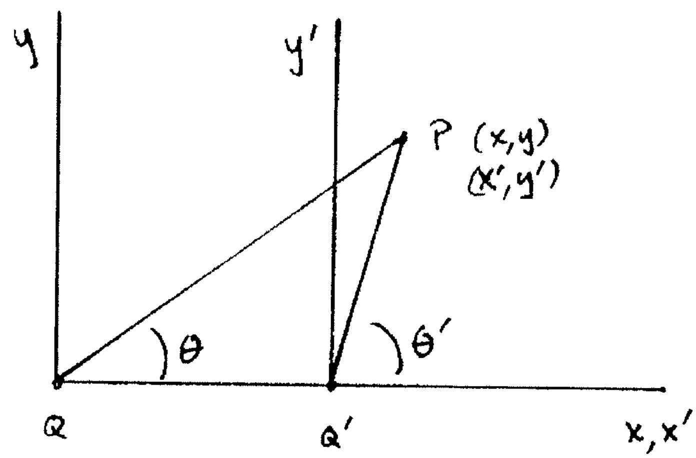
$$(1) \qquad \boldsymbol{E}_1 = E_{10} \sin(kz - \omega t)\,\hat{x}$$
$$(2) \qquad \boldsymbol{E}_2 = E_{20} \sin(kz - \omega t)\,\hat{y}$$
$$(3) \qquad \boldsymbol{E}_3 = E_{30} \sin(kz + \omega t)\,\hat{x}$$

with $k = 2\pi/\lambda$, $\omega = 2\pi/T = 2\pi f$, $\lambda f = c$, and c is the speed of light.

(a) Find the corresponding magnetic fields $\boldsymbol{B}$ for each wave.
(b) Calculate the Poynting Vector **S** for a superposition waves (1) and (2). Calculate also the time average of **S** for one period of oscillation.
(c) Calculate the Poynting Vector **S** for a superposition waves (1) and (3). Calculate also time average of **S** for one period of oscillation.
(d) Calculate the energy density of the electromagnetic wave resulting from a superposition of waves (1) and (3).
(e) Consider the results of (b) and (c), (d). Which of the two cases describes a standing or a traveling wave?

[I-10] **[10]**

Frame S' moves relative to S in the x direction with speed v. A light source Q' at rest at the origin in S' emits a spherical wave. The light wave is observed at the point P = (x,y) in the reference frame S and it is interpreted in that frame to have been emitted at the origin of S, located at Q in the figure below. This same point P has coordinates (x',y') in S' and an observer in S' simultaneously observes the light wave at that point. Derive the relationship between the angles in the two frames.
Hints: The light wave is spherical in either frame. It travels a distance r from its source Q at the origin in the S frame and a distance r' from its source Q' at the origin in the S' frame. The coordinates r,r' and t,t' are related by a Lorentz transformation.

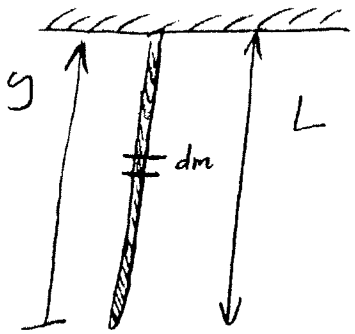


Solutions

I-1

I-1 [10]
A uniform rope of mass M and length L is hanging from a ceiling. The gravitational acceleration is g.

How long does it take for a transverse wave to traverse the full length of the rope?



tension must depend on the position:

$$dm = \frac{M}{L}\, dy$$

$$T(y+dy) - T(y) - g\frac{M}{L}\, dy = 0$$

$$\frac{dT}{dy} = g\frac{M}{L} \qquad \text{with} \quad T(0) = 0$$

$$\Longrightarrow \quad T(y) = \frac{Mg}{L}\, y$$

consequently, the propagation speed will also depend on y:

$$v(y) = \sqrt{\frac{T(y)}{\mu}} = \sqrt{\frac{\frac{Mg\,y}{L}}{M/L}} = \sqrt{gy}$$

Finally,

$$T = \int_0^T dt = \int_0^L \frac{dy}{v(y)} = \int_0^L \frac{dy}{\sqrt{gy}} = \frac{1}{\sqrt{g}}\int_0^L y^{-1/2}\, dy = \frac{2}{\sqrt{g}}\, y^{1/2}\Big|_0^L$$

$$= \boxed{2\sqrt{\frac{L}{g}}}$$

I-2

Solution: (I - 2)

(a) m_1 must have a velocity v perpendicular to the string such that the centripetal force on it is equal to the gravitational force on m_2:

$$\frac{m_1 v^2}{l-d} = m_2 g \ ,$$

or

$$v = \sqrt{\frac{m_2(l-d)g}{m_1}} \ .$$

(b) Use a frame of polar coordinates fixed in the horizontal table as shown in Fig. m_2 has z-coordinate $-(l-r)$ and thus velocity $\dot{r}$. The Lagrangian of the system is then

$$L = T - V = \frac{1}{2}m_1(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{1}{2}m_2\dot{r}^2 + m_2 g(l-r) \ .$$

Lagrange's equations give

$$m_1 r^2 \dot{\theta} = \text{constant} \ ,$$

$$(m_1 + m_2)\ddot{r} - m_1 r\dot{\theta}^2 + m_2 g = 0 \ .$$

At $t=0$, $r = l-d$, $v = \sqrt{m_2(l-d)g/m_1} = v_0$, say, so

$$\dot{\theta}_0 = \frac{v_0}{l-d} = \sqrt{\frac{m_2}{m_1}\frac{g}{l-d}} \ .$$

Hence

$$m_1 r^2\dot{\theta} = m_1(l-d)^2\dot{\theta}_0 = m_1\sqrt{\frac{m_2}{m_1}(l-d)^3 g} \ ,$$

giving

$$r\dot{\theta}^2 = \frac{r^4\dot{\theta}^2}{r^3} = \frac{m_2}{m_1}\left(\frac{l-d}{r}\right)^3 g$$

and

$$(m_1+m_2)\ddot{r} - m_2\left(\frac{l-d}{r}\right)^3 g + m_2 g = 0 \ .$$

Let $r = (l-d) + \rho$, where $\rho \ll (l-d)$. Then

$$\ddot{r} = \ddot{\rho}, \qquad r^{-3} = (l-d)^{-3}\left(1 + \frac{\rho}{l-d}\right)^{-3} \approx (l-d)^{-3}\left(1 - \frac{3\rho}{l-d}\right)$$

and the above equation becomes

$$\ddot{\rho} + \frac{3m_2 g}{(m_1+m_2)(l-d)}\rho = 0 \ .$$

Hence ρ oscillates about O, i.e. r oscillates about the value $l-d$, with angular frequency

$$\omega = \sqrt{\frac{3m_2 g}{(m_1+m_2)(l-d)}} \ ,$$

or period

$$T = 2\pi\sqrt{\frac{(m_1+m_2)(l-d)}{3m_2 g}} \ .$$

$\boxed{I-3}$

Symon 12.3

$$V(x,y,z) = \exp\left(f(x,y,z)/a^2 \right)$$

where

$$f(x,y,z) = 5x^2 + 5y^2 + 8z^2 - 8z^2 - 8yz - 26ya - 8za$$

$$\partial_x V = \frac{1}{a^2}\partial_x f\, V, \quad \partial_y V = \frac{1}{a^2}\partial_y f\, V, \quad \partial_z V = \frac{1}{a^2}\partial_z f\, V$$

must all vanish at the min.

$$\partial_x f = 10x, \quad \partial_y f = 10y - 8z - 26a,$$

$$\partial_z f = 16z - 8y - 8a$$

These vanish at

$$(x_0, y_0, z_0) = (0, 5a, 3a).$$

Next define $x_1 = x$, $x_2 = y$, $x_3 = z$,

$$\tilde{x}_1 = x - x_0, \quad \tilde{x}_2 = y - y_0, \quad \tilde{x}_3 = z - z_0$$

$$V_{ij} = \left. \frac{\partial^2 V}{\partial x_i \partial x_j} \right|_{x_i = x_{i0}}$$

Then we can expand about the minimum:

$$V = V(x_0, y_0, z_0) + \frac{1}{2}\sum_{ij} V_{ij}\, \tilde{x}_i \tilde{x}_j + \mathcal{O}(\tilde{x})^3$$

The equations of motion are:

$$m\ddot{\tilde{x}}_i = -\sum_j V_{ij}\tilde{x}_j$$

V_{ij} is a real symmetric matrix so there exists an orthogonal matrix R such that

$$\sum_{jk} R_{ij} V_{jk} R^T_{kl} = \delta_{il} k_i$$

Recall that $\sum_j R^T_{ij} R_{jk} = \delta_{ik}$.

Then in the equations of motion we have

$$\sum_j m R_{ij} \ddot{\tilde{x}}_j = - \sum_{\substack{jk \\ lm}} R_{ij} V_{jk} R^T_{kl} R_{lm} \tilde{x}_m$$

$$= - \sum_{lm} \delta_{il} k_i R_{lm} \tilde{x}_m$$

$$= - k_i \sum_j R_{ij} \tilde{x}_j$$

Now suppose

$$\sum_j R_{ij} \ddot{\tilde{x}}_j = - \omega_i^2 \sum_j R_{ij} \tilde{x}_j$$

We find

$$m \omega_i^2 = k_i \quad \Rightarrow \quad \boxed{\omega_i^2 = \frac{k_i}{m}}$$

All that remains is to find V_{ij} and its eigenvalues k_i.

Let $$f_{ij} = \frac{\partial^2}{\partial x_i \partial x_j} f \Big|_{x_i = x_{i0}} .$$

Then we compute:

$f_{xx} = 10, \quad f_{xy} = 0, \quad f_{xz} = 0$

$f_{yx} = 0, \quad f_{yy} = 10, \quad f_{yz} = -8$

$f_{zx} = 0, \quad f_{zy} = -8, \quad f_{zz} = 16$

$$[V_{ij}] = \frac{1}{a^2} V_0 \begin{pmatrix} 10 & 0 & 0 \\ 0 & 10 & -8 \\ 0 & -8 & 16 \end{pmatrix}$$

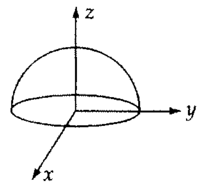
$$\begin{vmatrix} 10-\lambda & -8 \\ -8 & 16-\lambda \end{vmatrix} = 160 - 26\lambda + \lambda^2 - 64$$

$$\lambda_\pm = 13 \pm \sqrt{73}$$

$$\boxed{k_1 = \frac{10 V_0}{a^2}, \quad k_{2,3} = \frac{(13 \pm \sqrt{73})}{a^2} V_0}$$

I-4

Solution to I-4 Rotational Motion



Let the surface of the hemisphere lie in the x-y plane as shown. The mass density is given by

$$\rho = \frac{M}{V} = \frac{M}{\frac{2}{3}\pi b^3} = \frac{3M}{2\pi b^3}$$

First, we calculate the center of mass of the hemisphere. By symmetry

$$x_{\mathrm{CM}} = y_{\mathrm{CM}} = 0$$

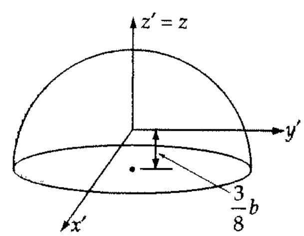
$$z_{\mathrm{CM}} = \frac{1}{M}\int_v \rho z \, dv$$

Using spherical coordinates $(z = r\cos\theta, dv = r^2 \sin\theta \, dr \, d\theta \, d\phi)$ we have

$$z_{\mathrm{CM}} = \frac{\rho}{M}\int_{\phi=0}^{2\pi} d\phi \int_{\theta=0}^{\pi/2} \sin\theta\cos\theta \, d\theta \int_{r=0}^{b} r^3 \, dr$$

$$= \left[\frac{3}{2\pi b^3}\right](2\pi)\left[\frac{1}{2}\right]\left[\frac{1}{4}b^4\right] = \frac{3}{8}b$$

We now calculate the inertia tensor with respect to axes passing through the center of mass:



By symmetry, $I_{12} = I_{21} = I_{13} = I_{31} = I_{23} = I_{32} = 0$. Thus the axes shown are the principal axes. Also, by symmetry $I_{11} = I_{22}$. We calculate I_{11} using Eq. 11.49:

$$I_{11} = J_{11} - M\left[\frac{3}{8}v\right]^2 \tag{1}$$

where J_{11} = the moment of inertia with respect to the original axes

$$J_{11} = \rho \int_v \left(y^2 + z^2\right) dv$$

$$= \int_v \left(r^2 \sin^2 \theta \sin^2 \phi + r^2 \cos^2 \theta\right) r^2 \sin\theta \, dr \, d\theta \, d\phi$$

$$= \frac{3M}{2\pi b^3} \int_{r=0}^{b} r^4 \, dr \int_{\theta=0}^{\pi/2} \left[\int_{\phi=0}^{2\pi} \left(\sin^2 \theta \sin^2 \phi + \cos^2 \theta\right) d\phi \right] \sin\theta \, d\theta$$

$$= \frac{3Mb^2}{10\pi} \int_{\theta=0}^{\pi/2} \left(\pi \sin^3 \theta + 2\pi \cos^2 \theta \sin\theta\right) d\theta$$

$$= \frac{2}{5} Mb^2$$

Thus, from (1)

$$I_{11} = I_{22} = \frac{2}{5} Mb^2 - \frac{9}{64} Mb^2 = \frac{83}{320} Mb^2$$

Also, from Eq. 11.49

$$I_{33} = J_{33} - M(0) = J_{33}$$

($I_{33} = J_{33}$ should be obvious physically)

So

$$I_{33} = \rho \int_v \left(x^2 + y^2\right) dv$$

$$= \rho \int_v r^4 \sin^3 \theta \, dr \, d\theta \, d\phi = \frac{2}{5} Mb^2$$

Thus, the principal axes are the primed axes shown in the figure. The principal moments of inertia are $$I_{11} = I_{22} = \frac{83}{320} Mb^2$$ $$I_{33} = \frac{2}{5} Mb^2$$

I-5

Answer

I-5 [(a) 3, (b) 3, (c)4] -Relativity (Mechanics):

(a) $E_{rest} = mc^2 = 9.1\times10^{-31}\times\left(3\times10^8\right)^2 = 8.19\times10^{-14}\left[J\right] = 512\left[keV\right]$

(b) $E_{total} = \dfrac{mc^2}{\sqrt{1-\dfrac{v^2}{c^2}}} = 2mc^2 \rightarrow \dfrac{1}{\sqrt{1-\dfrac{v^2}{c^2}}} = 2$

Solving for v, $v = \dfrac{\sqrt{3}}{2}c = 2.6\times10^8\left[m/s\right]$

(c) $K = E_{total} - mc^2 = 2mc^2 - mc^2 = mc^2 = 512\left[keV\right]$

I-7

I-7 [(a) 1, (b) 1, (c) 8] -Maxwell equation

(a)

$$div\mathbf{E} = \frac{\rho}{\varepsilon_0}$$

$$div\mathbf{B} = 0$$

$$rot\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}$$

$$rot\mathbf{B} = \mu_0\mathbf{j} + \mu_0\varepsilon_0\frac{\partial\mathbf{E}}{\partial t}$$

(b) $rot\mathbf{B} = \mu_0\mathbf{j} + \mu_0\varepsilon_0\dfrac{\partial\mathbf{E}}{\partial t}$

(c) Take div of the forth equation. $divrot\mathbf{B} = 0 = \mu_0 div\mathbf{j} + \mu_0\varepsilon_0 div\dfrac{\partial\mathbf{E}}{\partial t}$, I-7 cont'd

Take time derivative of the first equation. $\frac{\partial}{\partial t} div\mathbf{E} = div\frac{\partial}{\partial t}\mathbf{E} = \frac{1}{\varepsilon_0}\frac{\partial\rho}{\partial t}$

$\rightarrow 0 = \mu_0 div\mathbf{j} + \mu_0\varepsilon_0 div\frac{\partial\mathbf{E}}{\partial t} = \mu_0 div\mathbf{j} + \mu_0\frac{\partial\rho}{\partial t} \rightarrow div\mathbf{j} + \frac{\partial\rho}{\partial t} = 0$

If the displacement current term, $\mu_0\varepsilon_0\frac{\partial\mathbf{E}}{\partial t}$ does not exists, it becomes $div\mathbf{j} = 0$ and does not satisfy the continuity equation.

II-3 [10] -Spin/angular momentum:

$\hat{S}^2 = (\hat{\mathbf{S}}_1 + \hat{\mathbf{S}}_2)^2 = \hat{S}_1^2 + \hat{S}_2^2 + 2\hat{\mathbf{S}}_1\cdot\hat{\mathbf{S}}_2$

$\rightarrow \hat{\mathbf{S}}_1\cdot\hat{\mathbf{S}}_2 = \frac{1}{2}(\hat{S}^2 - \hat{S}_1^2 - \hat{S}_2^2)$

Eigen value equation therefore be written as

$$\hat{H}|s,n\rangle = \lambda\hat{\mathbf{S}}_1\cdot\hat{\mathbf{S}}_2|s,n\rangle = \frac{\lambda}{2}(\hat{S}^2 - \hat{S}_1^2 - \hat{S}_2^2)|s,n\rangle = \frac{\lambda\hbar^2}{2}\left(s(s+1) - \frac{1}{2}\left(\frac{1}{2}+1\right) - \frac{1}{2}\left(\frac{1}{2}+1\right)\right)|s,n\rangle$$

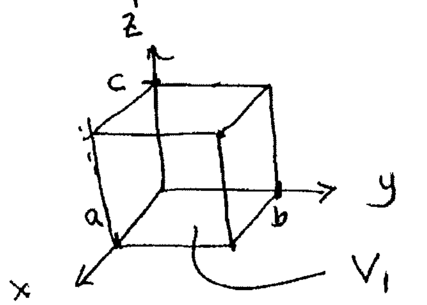
$$= \frac{\lambda\hbar^2}{2}\left(s(s+1) - \frac{3}{2}\right)|s,n\rangle$$

For triplet state (s=1), $E = \frac{\lambda\hbar^2}{2}\left(1(1+1) - \frac{3}{2}\right) = \frac{\lambda\hbar^2}{4}$

For singlet state(s=0), $E = \frac{\lambda\hbar^2}{2}\left(0 - \frac{3}{2}\right) = -\frac{3\lambda\hbar^2}{4}$

$\boxed{I-6}$ 1. Potential with conducting box, except one side.

(a) Let's assume all surfaces have zero potential, except the $z=0$ surface



$$\phi(x,y,z) \sim e^{\pm i\alpha x} e^{\pm i\beta y} e^{\pm \gamma z} \qquad \nabla^2\phi = 0$$

$$\sim X(x)\, Y(x)\, Z(z)$$

$$X(x) = \sin\left(\frac{r\pi x}{a}\right)$$

$$Y(y) = \sin\left(\frac{s\pi x}{b}\right)$$

$$Z(z) = \sinh\left(\gamma_{rs}(c-z)\right) \qquad \gamma_{rs} = \pi\sqrt{\frac{r^2}{a^2} + \frac{s^2}{b^2}}$$

$$\phi_{rs} = \sin\left(\frac{r\pi x}{a}\right)\sin\left(\frac{s\pi x}{b}\right)\sinh\left(\gamma(c-z)\right)$$

$$\phi(x,y,z) = \sum_{r,s} A_{rs}\phi_{rs}$$ where A_{rs} are determined by the boundary condition at $z=0$

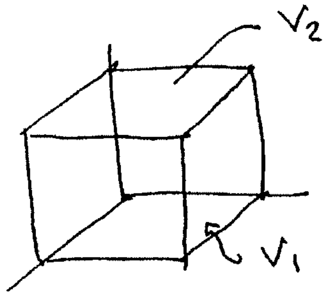
$$V_1 = \sum_{r,s=1}^{\infty} A_{rs} \sin\left(\frac{r\pi x}{a}\right)\sin\left(\frac{s\pi y}{b}\right)\sinh(\gamma_{rs}c)$$

Invert the double sine series and get

$$A_{rs} = \frac{4}{\pi r}\cdot\frac{4}{\pi s}\cdot\frac{V_1}{\sinh(\gamma_{rs}c)}$$

$$\phi_1(x,y,z) = \frac{16}{\pi^2} V_1 \sum_{\substack{r,s=1\\ \text{odd}}}^{\infty} \frac{1}{rs}\frac{\sinh(\gamma_{rs}(c-z))}{\sinh(\gamma_{rs}c)}\cdot\sin\left(\frac{r\pi x}{a}\right)\cdot\sin\left(\frac{s\pi y}{b}\right)$$

(b)



Solve the problem with $V_2 \neq 0$ and $V_1 = 0$

In analogy with part a, this is

$$\phi_2(x,y,z) = \frac{16}{\pi^2} V_2 \sum_{\substack{r,s=1 \\ \text{odd}}}^{\infty} \frac{1}{rs} \frac{\sinh(\gamma_{rs} z)}{\sinh(\gamma_{rs} c)} \cdot \sin\left(\frac{r\pi x}{a}\right) \sin\left(\frac{s\pi y}{b}\right)$$

Then, the potential is the sum of ϕ_1 and ϕ_2

$$\phi(x,y,z) = \phi_1(x,y,z) + \phi_2(x,y,z)$$

$$= \frac{16}{\pi^2} \sum_{\substack{r,s \\ \text{odd}}} \frac{1}{rs} \left(\frac{V_1 \sinh(\gamma_{rs}(c-z)) + V_2 \sinh(\gamma_{rs} z)}{\sinh(\gamma_{rs} c)} \right) \cdot \sin\left(\frac{r\pi x}{a}\right) \sin\left(\frac{s\pi y}{b}\right)$$

I-8

Solutions I-8.

The magnetic force acting on the loop is proportional to its magnetic moment, which is proportional to the current flowing through the loop. The current I, in turn, is proportional to the rate of change of the magnetic flux through the loop, since $I = \mathcal{E}_e/R$, where $\mathcal{E}_e$ is the electromotive force and R is the resistance of the loop. We have

$$\mathcal{E}_e = -\frac{1}{c}\frac{d\Phi}{dt} \tag{1}$$

The magnetic flux Φ in 1) is given by

$$\Phi = BS = B_0(1+\kappa z)S \tag{2}$$

where S is the area of the loop. From (S.3.36.1),

$$\mathcal{E}_e = -\frac{1}{c}\frac{d\Phi}{dt} = -\frac{1}{c}B_0\kappa\frac{dz}{dt}S \tag{3}$$

But dz/dt is the velocity of the loop. So the electromotive force increases with the velocity, and therefore the magnetic force F_m acting on the loop also increases with velocity, while the only other force, gravity, acting in the opposite direction, is constant. Therefore, the velocity will increase until $F_m = mg$. From energy conservation, the work done by gravity during this stationary motion goes into the Joule heating of the loop:

$$mg\Delta z = I^2R\Delta t \tag{4}$$

But, since the velocity is constant,

$$\frac{\Delta z}{\Delta t} = v = \frac{I^2R}{mg} = \frac{\mathcal{E}_e^2}{Rmg} = \frac{B_0^2}{c^2}\frac{\kappa^2v^2S^2}{Rmg} \tag{5}$$

where we substituted $\mathcal{E}_e$ from .3) again using $v = dz/dt$. From 5), we can find

$$v = \frac{c^2Rmg}{B_0^2\kappa^2S^2} \tag{6}$$

Now, substituting

$$R = \rho\frac{\pi D}{\pi d^2/4} = \rho\frac{4D}{d^2} \tag{7}$$

and

$$m = \rho_m V = \rho_m\frac{\pi d^2}{4}\pi D \tag{8}$$

into 6), we obtain

$$v = \frac{c^2\rho\rho_m g\pi^2D^2}{B_0^2\kappa^2(\pi D^2/4)^2} = \frac{16c^2\rho\rho_m g}{B_0^2\kappa^2D^2}$$

I-9

I-9 Solution

(a) $\vec{B} = \int \vec{\nabla} \times \vec{E}\, dt$

(1) $\vec{B}_1 = \frac{kE_{10}}{\omega} \sin(kz - \omega t)\,\hat{y}$

(2) $\vec{B}_2 = -\frac{kE_{20}}{\omega} \sin(kz - \omega t)\,\hat{x}$

(3) $\vec{B}_3 = -\frac{kE_{30}}{\omega} \sin(kz + \omega t)\,\hat{y}$

(b) $\vec{E}' = \vec{E}_1 + \vec{E}_2$, $\vec{B}' = \vec{B}_1 + \vec{B}_2$, $\vec{S}' = \frac{1}{\mu_0} \vec{E}' \times \vec{B}'$, $\langle \vec{S} \rangle = \int_0^T \vec{S}' dt$

$T = \frac{\omega}{2\pi}$

$\vec{S}' = \frac{k}{\omega \mu_0} \sin^2(kz - \omega t)(E_{10}^2 + E_{20}^2)\,\hat{z}$

$\langle \vec{S}' \rangle = \frac{1}{2} \frac{E_1^2 + E_2^2}{\mu_0 c}\,\hat{z}$

(c) $\vec{E}''' = \vec{E}_1 + \vec{E}_3$, $\vec{B}''' = \vec{B}_1 + \vec{B}_3$

$\vec{S}''' = -\frac{4E_1^2}{\mu_0 c} \sin kz \cos kz \sin \omega t \cos \omega t\,\hat{z}$

$\langle \vec{S}''' \rangle = 0$

(d) $W = \frac{1}{2}(\varepsilon_0 \vec{E}'''^2 + \frac{1}{\mu_0} \vec{B}'''^2)$

$W = 2\varepsilon_0 E_1^2 (\sin^2 kz \cos \omega t + \cos^2 kz \sin \omega t)$

(e) (b) traveling wave

(c), (d) standing wave

I-10

Problem I-10
From Wangsness pp. 504-506

The spherical wave in the two frames is described by

$$\psi' = \psi_0' \frac{1}{r'} e^{i(k'r' - \omega' t')}$$

$$\psi = \psi_0 \frac{1}{r} e^{i(kr - \omega t)}$$

They must have the same complex phase $\Rightarrow$

$$k'r' - \omega' t' = kr - \omega t \qquad (*)$$

But $k' = \omega'/c$, $k = \omega/c$

$$r' = x' \cos\theta' + y' \sin\theta'$$

$$r = x \cos\theta + y \sin\theta$$

$$x' = \gamma(x - \beta c t), \quad y' = y$$

$$t' = \gamma\left(t - \frac{\beta}{c} x\right)$$

Substitute these into (*) :

Physics PhD Qualifying Examination
Part II – Friday, August 27, 2010

Name: ______________________________________
(please print)
Identification Number: ________

STUDENT: insert a check mark in the left boxes to designate the problem numbers that you are handing in for grading.
PROCTOR: check off the right hand boxes corresponding to the problems received from each student. Initial in the right hand box.

	1	
	2	
	3	
	4	
	5	
	6	
	7	
	8	
	9	
	10	

Student's initials
problems handed in:
Proctor's initials

INSTRUCTIONS FOR SUBMITTING ANSWER SHEETS

1. DO NOT PUT YOUR NAME ON ANY ANSWER SHEET. EXAMS WILL BE COLLATED AND GRADED BY THE ID NUMBER ABOVE.
2. Use at least one separate preprinted answer sheet for each problem. Write on only one side of each answer sheet.
3. Write your **identification number** listed above, in the appropriate box on the preprinted sheets.
4. Write the **problem number** in the appropriate box of each preprinted answer sheet. If you use more than one page for an answer, then number the answer sheets with both problem number and page (e.g. Problem 9 – Page 1 of 3).
5. Staple together all the pages pertaining to a given problem. Use a paper clip to group together all eight problems that you are handing in.
6. Hand in a total of *eight* problems. A passing distribution will normally include at least four passed problems from problems 1-6 (Quantum Physics) and two problems from problems 7-10 (Thermodynamics and Statistical Mechanics). **DO NOT HAND IN MORE THAN EIGHT PROBLEMS.**
7. **YOU MUST SHOW ALL YOUR WORK.**

[II-4] **[5, 5]**

Consider the scattering of a particle from a three-dimensional "square well" potential,

$$V(r) = \begin{cases} -V_0 & \text{for} \quad r \le a \\ 0 & \text{for} \quad r > a \end{cases} .$$

(a) In the Born approximation, compute the differential cross section for an incident plane wave.
(b) Compute the total cross section for arbitrary k.

[II-5] **[10]**

A and B are hermitian operators on the complex vector space V. The commutator of two operators is defined as $[\,A\,,B\,] = A\,B - B\,A$.

Prove that

$$\|\, A\,\Phi \,\|\; \|\, B\,\Phi \,\| \;\ge\; \tfrac{1}{2}\,|\,(\,\Phi,\, [A,B]\,\Phi)\,|$$

for any vector Φ being an element of V, where $\|\ldots\|$ and $(\ldots,\ldots)$ denote the norm and the scalar product in vector space V, respectively.

Hints:

- Express $|\,(\,\Phi,\,[\,A\,,\,B\,]\,\Phi)\,|$ using $\mathrm{Im}\,(\,A\Phi,\, B\Phi\,)$.
- $|\mathrm{Im}\,(\chi,\Phi)| \le |(\chi,\Phi)|$.
- Schwartz's Inequality: $|\,(A\Phi,B\Phi)\,| \le \|\,A\,\Phi\,\|\;\|\,B\,\Phi\,\|$.

(Explanations: Im=imaginary part, χ is an element of V.)

[II-6]　　[1,1,1,7]

A static magnetic field, $(0, 0, B_0)$ and a rotating magnetic field $(B_1 \cos\omega t, B_1 \sin\omega t, 0)$ are applied to a particle with ½ spin. Both of the fields are uniform throughout space. The Hamiltonian of the system is given by the following equation:

$$H(t) = H_0 + H_1(t) \text{ with } H_0 = -\frac{1}{2}\gamma\hbar B_0 \sigma_z \text{ and } H_1(t) = -\frac{1}{2}\gamma\hbar B_1\left(\sigma_x \cos\omega t + \sigma_y \sin\omega t\right),$$

where γ is the gyromagnetic ratio of the particle, and $\sigma_x, \sigma_y, \sigma_z$ are the operators for the x, y, z components of the spin.

The general spin state of the system is given by $|\Psi(t)\rangle = c_1(t)|1\rangle e^{-i\omega_1 t} + c_2(t)|2\rangle e^{-i\omega_2 t}$ where $|1\rangle$ and $|2\rangle$ are the eigenstates of H_0 corresponding to the up and down spins, respectively. $\omega_1 = -\frac{\gamma B_0}{2}$ and $\omega_2 = \frac{\gamma B_0}{2}$. Assume $B_0 >> B_1$ and that the spin of the system was pointing down at $t = 0$.

(a) Write down the 2x2 matrix elements of $H_1(t)$.

(b) Find the differential equation for $c_1(t)$ which determines its time evolutions.

(c) Using first order perturbation theory, find the 0^{th} and the 1^{st} order coefficients for $c_1(t) = c_1^{(0)} + c_1^{(1)}(t)$.

(d) Find the transition probability from down- to up-spin states at time t.

[II-7] **[6,4]**

(a) Derive the Clausius-Clapeyron equation for the equilibrium of two phases of a substance. Consider a liquid or solid phase in equilibrium with its vapor.

(b) Using part (a) and the ideal gas law for the vapor phase, show that the vapor pressure follows the equation $\ln(P_v) = A - B/(k_B T)$, where k_B is the Boltzmann constant and T is temperature. Make reasonable assumptions as required. What is B?

Hint: The Clausius-Clapeyron equation is an expression for dP/dT, which is the slope of the equilibrium line between the two phases (either liquid or solid phase in equilibrium with the vapor phase).

[II-8] **[10]**

The chemical potential of a single-component gas is given by

$$\mu(T,P) = -RT\ln\left(a\frac{T^{5/2}}{(P+b)}\right),$$

where a, b are material-specific positive constants.

Obtain the internal energy of the system $u(T,v)$.

[II-9] [7,2,1]

Consider a system of N localized weakly interacting particles, each with spin ½ and magnetic moment μ, located in an external magnetic field H. The energy of the system is

$$E = -(n_1 - n_2)\mu H ,$$

where n_1 is the number of spins aligned parallel to H and n_2 is the number of spins aligned antiparallel to H.

(a) Write an expression for the entropy, $S(E)$, for this system.
(b) Find the temperature of the system as a function of E.
(c) Which is hotter – a system with positive T or a system with a negative T? State in your answer what is your definition of hot vs. cold in terms of the laws of thermodynamics.

Use Stirling's approximation: $\ln(n!) \approx n\ln(n) - n$ for large n.

[II-10] [10]

Consider a *two-dimensional photon gas* confined to an area $A = L \times L$. What is the average number of photons in the system at temperature T?
Your answer must be expressed in terms of A, T, and of course, the necessary fundamental constants.

To keep your expressions relatively compact, the following relationship will come useful:
$\int_0^\infty \frac{x^{\nu-1}}{e^x - 1}dx = \Gamma(\nu)\zeta(\nu)$, where $\Gamma(\nu)$ is the gamma function and $\zeta(\nu)$ is the Rieman zeta function. In particular, $\Gamma(2) = 1! = 1$ and $\zeta(2) = \frac{\pi^2}{6}$.
(You must *derive* your answer. Guessing or hand-waving the answer, or pulling it out of your memory, will yield zero credit.)

Problem II-1

[redacted]

Let L denote $x<-a$, R denote $x>a$.

<u>flux</u>

$$j = \frac{\hbar}{2im}\left(\psi^* \frac{d\psi}{dx} - \frac{d\psi^*}{dx}\psi\right)$$

$$j_L = \frac{\hbar}{2im}\Big[\left(Ae^{ikx}+Be^{-ikx}\right)^*\left(ikAe^{ikx}-ikBe^{-ikx}\right) - c.c.\Big]$$

$$= \frac{\hbar}{2im}\Big[ik|A|^2 - ik|B|^2 - ikBA^*e^{-2ikx} + ikAB^*e^{2ikx} - c.c.\Big]$$

$$= \frac{\hbar}{2im}\Big[2ik|A|^2 - 2ik|B|^2 - ikBA^*e^{-2ikx} - ikB^*Ae^{2ikx} + ikAB^*e^{2ikx} + ikA^*Be^{-2ikx}\Big]$$

$$= \frac{\hbar k}{m}\left(|A|^2 - |B|^2\right)$$

$$j_R = \frac{\hbar k}{m}\left(|C|^2 - |D|^2\right)$$

Flux conservation is

$$|A|^2 - |B|^2 = |C|^2 - |D|^2$$

$$= |S_{11}A + S_{12}D|^2 - |D|^2$$

$$= |A|^2 - |S_{21}A + S_{22}D|^2$$

$$= |S_{11}|^2|A|^2 + |S_{12}|^2|D|^2 + S_{11}S_{12}^* A D^* + S_{11}^* S_{12} A^* D - |D|^2$$

$$= |A|^2 - |S_{21}|^2|A|^2 - |S_{22}|^2|D|^2 - S_{21}S_{22}^* A D^* - S_{21}^* S_{22} A^* D$$

$$0 = \left(|S_{11}|^2 + |S_{21}|^2 - 1\right)|A|^2 + \left(|S_{12}|^2 + |S_{22}|^2 - 1\right)|D|^2 + \left(S_{11}S_{12}^* + S_{21}S_{22}^*\right)AD^* + \left(S_{11}^*S_{12} + S_{21}^*S_{22}\right)A^*D$$

$\Rightarrow$

$$|S_{11}|^2 + |S_{21}|^2 = 1$$

$$|S_{12}|^2 + |S_{22}|^2 = 1$$

$$S_{11}S_{12}^* + S_{21}S_{22}^* = 0$$

$\boxed{\text{II}-2}$

Solution II-2 (I have to check 1 more time the result of the Final integration + units. IW)

(a) potential energy $r > r_0$ $\quad U(r) = -\dfrac{e^2}{4\pi\varepsilon_0 r}$

potential energy $\underline{r \le r_0}$ $\quad U(r) = \dfrac{e^2}{8\pi\varepsilon_0 r_0}\left(\left(\dfrac{r}{r_0}\right)^2 - 3\right)$

Gauss' law $\quad E = \dfrac{er}{4\pi\varepsilon_0 r_0^3} \quad r \le r_0$

$$\phi(r) = \phi(r_0) + \frac{e}{4\pi\varepsilon_0 r_0^3}\int_r^{r_0} r\,dr$$

$$\phi(r) = \frac{e}{4\pi\varepsilon_0 r_0} + \frac{e(r_0^2 - r^2)}{8\pi\varepsilon_0 r_0^3}$$

$$\underline{U(r) = -e\phi(r)} = \frac{e^2}{8\pi\varepsilon_0 r_0}\left(\left(\frac{r}{r_0}\right)^2 - 3\right)$$

(b) $r > r_0 \quad H = \dfrac{p^2}{2\mu} - \dfrac{e^2}{4\pi\varepsilon_0 r} = H_0$

$r \le r_0 \quad H = \dfrac{p^2}{2\mu} + \dfrac{e^2}{8\pi\varepsilon_0 r_0}\left(\left(\dfrac{r}{r_0}\right)^2 - 3\right) = H_0 + H'$

$$\Updownarrow \; H' = H_0 + H' - H_0$$

$$= \frac{p^2}{2\mu} + \frac{e^2}{8\pi\varepsilon_0 r_0}\left(\left(\frac{r}{r_0}\right)^2 - 3\right) - \frac{p^2}{2\mu} - \frac{e^2}{4\pi\varepsilon_0 r}$$

$$= \frac{e^2}{4\pi\varepsilon_0}\left(\frac{1}{2r_0}\left(\left(\frac{r}{r_0}\right)^2 - 3\right) - \frac{1}{r}\right)$$

(c) $E_0^1 = \langle 100 | H' | 100 \rangle$

$$= \int d\Omega \int_0^{r_0} r^2 \; |R_{10}(r)|^2 \, H'(r)\, dr$$

$$E_0^1 = \frac{1}{4\pi\varepsilon_0}\left(\frac{e^2}{4\pi}\left(\frac{3a_0^3 - 3a_0 r_0^2 + 3r_0^3 - 3a_0 e^{-2R/a_0}(a_0+R)^2}{2R^3 a_0}\right)\right)$$

Take time derivative of the first equation. $\frac{\partial}{\partial t} div\mathbf{E} = div\frac{\partial}{\partial t}\mathbf{E} = \frac{1}{\varepsilon_0}\frac{\partial \rho}{\partial t}$

$\rightarrow 0 = \mu_0 div\mathbf{j} + \mu_0\varepsilon_0 div\frac{\partial \mathbf{E}}{\partial t} = \mu_0 div\mathbf{j} + \mu_0\frac{\partial \rho}{\partial t} \rightarrow div\mathbf{j} + \frac{\partial \rho}{\partial t} = 0$

If the displacement current term, $\mu_0\varepsilon_0\frac{\partial \mathbf{E}}{\partial t}$ does not exists, it becomes $div\mathbf{j} = 0$ and does not satisfy the continuity equation.

$\boxed{\text{II}-3}$

II-3 [10] -Spin/angular momentum:

$$\hat{S}^2 = \left(\hat{\mathbf{S}}_1 + \hat{\mathbf{S}}_2\right)^2 = \hat{S}_1^2 + \hat{S}_2^2 + 2\hat{\mathbf{S}}_1\cdot\hat{\mathbf{S}}_2$$

$$\rightarrow \hat{\mathbf{S}}_1\cdot\hat{\mathbf{S}}_2 = \frac{1}{2}\left(\hat{S}^2 - \hat{S}_1^2 - \hat{S}_2^2\right)$$

Eigen value equation therefore be written as

$$\hat{H}|s,n\rangle = \lambda\hat{\mathbf{S}}_1\cdot\hat{\mathbf{S}}_2|s,n\rangle = \frac{\lambda}{2}\left(\hat{S}^2 - \hat{S}_1^2 - \hat{S}_2^2\right)|s,n\rangle = \frac{\lambda\hbar^2}{2}\left(s(s+1) - \frac{1}{2}\left(\frac{1}{2}+1\right) - \frac{1}{2}\left(\frac{1}{2}+1\right)\right)|s,n\rangle$$

$$= \frac{\lambda\hbar^2}{2}\left(s(s+1) - \frac{3}{2}\right)|s,n\rangle$$

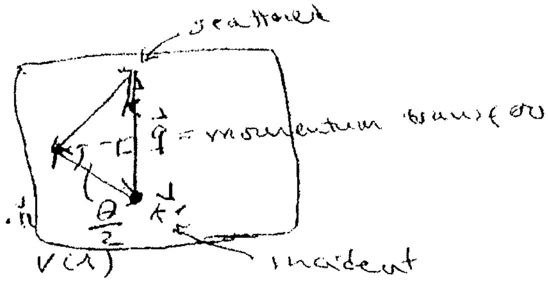
For triplet state (s=1), $E = \frac{\lambda\hbar^2}{2}\left(1(1+1) - \frac{3}{2}\right) = \frac{\lambda\hbar^2}{4}$

For singlet state(s=0), $E = \frac{\lambda\hbar^2}{2}\left(0 - \frac{3}{2}\right) = -\frac{3\lambda\hbar^2}{4}$

3. Born Approximation for "square well"

$$V(r) = \begin{cases} -V_0 & r \le a \\ 0 & r > a \end{cases}$$

$q = |k - k'| = 2k \sin\left(\frac{\theta}{2}\right)$
= momentum transfer



a) $$f(\Omega) = \frac{-m}{2\pi \hbar^2} \int \underbrace{d^3r}_{r^2\,dr\,d\Omega}\, e^{i(\vec{k}-\vec{k}')\cdot\vec{r}}$$

$$= \frac{-2m}{\hbar^2} \int_0^a r\,dr\, \frac{\sin(qr)}{q}$$

$$= \frac{2mV_0}{q^3 \hbar^2} \left[\sin(qa) - qa\cos(qa) \right]$$

$$= \frac{2mV_0 a^3}{(qa)^3 \hbar^2} \left[\sin(qa) - qa\cos(qa) \right]$$

$$\boxed{\frac{d\sigma}{d\Omega} = |f(\Omega)|^2 = \frac{4m^2 V_0^2 a^6}{\hbar^4 (qa)^6} \left(\sin(qa) - qa\cos(qa) \right)^2}$$

Plot vs θ for $ka = 1$

Let $$\frac{d\sigma}{d\Omega} = \frac{A_0}{x^6} \left(\sin(x) - x\cos(x) \right)^2$$

where $$A_0 = \frac{4m^2 V_0^2 a^6}{\hbar^4}$$

$$x = qa = 2ka\sin\left(\frac{\theta}{2}\right)$$
$$= 2\sin(\theta/2) \quad \text{for} \quad ka = 1.$$

<u>limits</u>:

$\theta \to 0 \qquad x \sim 0$

$$\lim_{x \to 0} \frac{A_0}{x^6} \left(\sin(x) - x\cos(x) \right)^2$$

$$\sim \lim_{x \to 0} \frac{1}{x^6} \left\{ \left(x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \right) - x\left(1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 \right) \right\}^2$$

$$\sim \lim_{x \to 0} \frac{1}{x^6} \left\{ 0 \cdot x + \left(-\frac{1}{3!} + \frac{1}{2!} \right) x^3 + \ldots \right\}^2 \sim \left(\frac{1}{3}\right)^2 = \frac{1}{9}$$

(3.2)

Minimum (or max)
let

$$\frac{d\sigma}{d\Omega} = \xi(x)$$

$$\frac{d\xi(x)}{d\theta} = \frac{d\xi(x)}{dx} \cdot \frac{dx}{d\theta} \qquad \frac{dx}{d\theta} = \cos(\theta/2)$$

$$\frac{d\xi}{dx} = \frac{A_0}{x^6} 2(\sin(x) - x\cos(x)) \cdot (\cancel{\cos(x)} - \cancel{\cos(x)} + x\sin(x))$$

$$-\frac{6A_0}{x^7}(\sin(x) - x\cos(x))^2$$

Need to approximate (numerically)

$$\frac{d\xi(x)}{d\theta} = 0 = \frac{\cos\left(\frac{\theta}{2}\right)}{x^6}\left\{ A_0 \left(2(\sin(x) - x\cos(x)\right) x\cdot\sin(x) - \frac{6A_0}{x}(\sin(x) - x\cos(x))^2 \right\}$$

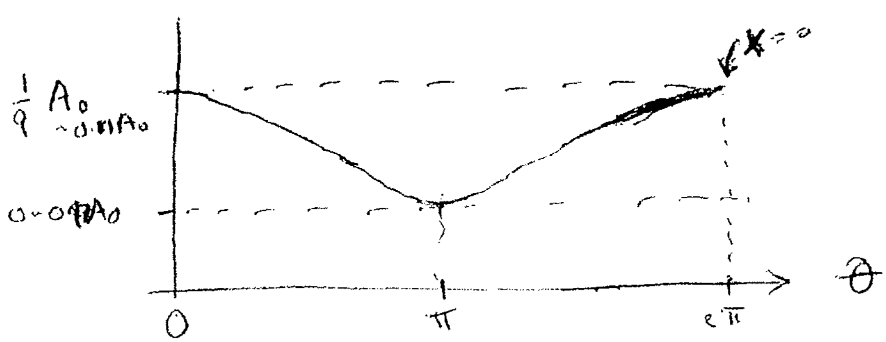
neglect second term vs 1st term

① $x\sin(x) = 0$ $x = 0$ (already done before). $x = \pi$ → out of range

② $\cos\left(\frac{\theta}{2}\right) = 0$ $\theta = \pi$. $x = 2\sin(\theta/2)$

$\theta = \pi$ is a minimum $\sin(\theta/2) = 1$

$$\xi(1) = \frac{A_0}{(2)^6}(\sin(2) - 2\cos(2))^2 \simeq A_0 \times 0.047$$



b)

Total cross section – for arbitrary k.

$$\sigma = \int_{4\pi} \frac{d\sigma}{d\Omega} \, d\Omega$$

$$= A_0 \underbrace{(2\pi)}_{\text{azimuthal symmetry}} \int_0^{\pi} \frac{d\theta \sin(\theta)}{(2ka \sin\frac{\theta}{2})^6} \quad , \quad A_0 = \frac{4m^2 V_0^2 a^6}{\hbar^4}$$

$$\times \left[2ka \sin\left(\tfrac{\theta}{2}\right) \cos\left(2ka\sin\left(\tfrac{\theta}{2}\right)\right) - \sin\left(2ka\sin\tfrac{\theta}{2}\right) \right]^2$$

Change of variables

$$\sin\theta = 2\cos\left(\tfrac{\theta}{2}\right)\sin\left(\tfrac{\theta}{2}\right)$$

$$= \frac{2\pi A_0}{\cancel{2}ka} \int_0^{\pi} \frac{(d\theta) \cdot \cancel{2} \cos\left(\frac{\theta}{2}\right) \cancel{\left(\sin\left(\frac{\theta}{2}\right) \cdot 2ka\right)}}{\left[2ka\sin\left(\frac{\theta}{2}\right)\right]^{\cancel{6}\,5}}$$

$$\times \left[2ka \sin\left(\tfrac{\theta}{2}\right) \cos\left(2ka\sin\tfrac{\theta}{2}\right) - \sin\left(2ka\sin\tfrac{\theta}{2}\right) \right]^2$$

$$2ka\sin\left(\tfrac{\theta}{2}\right) = x \qquad dx = ka\cos\left(\tfrac{\theta}{2}\right) d\theta$$

$$\theta = 0 \rightarrow x = 0$$

$$\theta = \pi \rightarrow x = 2ka$$

$$\sigma = \underbrace{\frac{2\pi A_0}{(ka)^2}}_{\frac{8\pi}{k^2}\left(\frac{m^2 V_0^2 a^4}{\hbar^4}\right) = A_1} \int_0^{2ka} \frac{dx}{x^5} \left(\sin(x) - x \cdot \cos(x) \right)^2$$

$$= A_1 \int_0^{2ka} \left(\frac{\sin^2 x}{x^5} - \frac{2\sin(x)\cos(x)}{x^4} + \frac{\cos^2(x)}{x^3} \right) dx$$

careful; each term may diverge, but sum does not!

$$= A_1 \int_0^{2ka} \left\{ \frac{-1}{4}\frac{d}{dx}\left(\frac{\sin^2 x}{x^4}\right) + \frac{1}{4}\frac{d}{dx}\left(\frac{\sin(2x)}{x^3}\right) + \frac{1}{4}\frac{d}{dx}\left(-\frac{1}{x^2}\right)\right\} dx$$

$$= \frac{1}{4} A_1 \left\{ -\left(\frac{\sin^2(x)}{x^4}\right) + \left(\frac{\sin(2x)}{x^3}\right) - \frac{1}{x^2} \right|_0^{2ka}$$

Evaluation at $x \to 0$ (check for divergence)

$$\lim_{x \to 0} -\frac{\sin^2(x)}{x^4} + \frac{\sin(2x)}{x^3} - \frac{1}{x^2}$$

$$= \lim_{x \to 0} \frac{-1}{x^4}\left(x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \dots\right)^2 + \frac{1}{x^3}\left(2x - \frac{1}{3!}(2x)^3 + \dots\right) - \frac{1}{x^2}$$

$$= \lim_{x \to 0} -\frac{1}{x^4}\left(x^2 - \frac{2}{3!}x^4 + \mathcal{O}(x^6)\right) + \frac{1}{x^3}\left(2x - \frac{8}{6}x^3 + \mathcal{O}(x^5)\right) - \frac{1}{x^2}$$

$$= \lim_{x \to 0} \underbrace{-\frac{1}{x^2} + \frac{2}{x^2} - \frac{1}{x^2}}_{0} + \left(\frac{1}{3} - \frac{8}{6}\right)(x)^0 + \mathcal{O}(x^2) = -1$$

$$\boxed{\sigma = \frac{1}{4} A_1 \left\{ 1 - \frac{\sin^2(2ka)}{(2ka)^4} + \frac{\sin(4ka)}{(2ka)^3} - \frac{1}{(2ka)^2} \right\}}$$

$$= \frac{2\pi}{k^2}\left(\frac{m^2 V_0^2 a^4}{\hbar^4}\right) \cdot \left\{ 1 - \frac{\sin^2(2ka)}{(2ka)^4} + \frac{\sin(4ka)}{(2ka)^3} - \frac{1}{(2ka)^2} \right\}$$

II-5

II-5 Solution

$A^* = A \quad , B^* = B$

$$\frac{1}{2} |(\phi, [A,B]\phi| = \frac{1}{2} | \phi, (AB-BA)\phi|$$

$$= \frac{1}{2} |(\phi, AB\phi) - (\phi, BA\phi)|$$

$$= \frac{1}{2} |(A\phi, B\phi) - (B\phi, A\phi)|$$

$$= \frac{1}{2} |(A\phi, B\phi) - \overline{(A\phi, B\phi)}|$$

$$= \frac{1}{2} |2 \operatorname{Im} (A\phi, B\phi)|$$

$$= \operatorname{Im} (A\phi, B\phi)$$

$$\leq |(A\phi, B\phi)|$$

$$\leq \|A\phi\| \, \|B\phi\|$$

$\boxed{\text{II}=6}$

[(a) 1, (b) 1, (c) 1, (d) 7] -Time-dependent perturbation:

(a) Assume eigen states of σ_z are given by the up spin, $|1\rangle$ and down spin $|2\rangle$ states, and corresponding eigen values are $E_1 = -\frac{1}{2}\gamma\hbar B_0$ and $E_2 = \frac{1}{2}\gamma\hbar B_0$.

$$\langle 1|H_1(t)|2\rangle = -\frac{1}{2}\gamma\hbar B_1(\cos\omega t + i\sin\omega t) = -\frac{1}{2}\gamma\hbar B_1 e^{i\omega t}$$

$$\langle 2|H_1(t)|1\rangle = -\frac{1}{2}\gamma\hbar B_1(\cos\omega t - i\sin\omega t) = -\frac{1}{2}\gamma\hbar B_1 e^{-i\omega t}$$

$$\langle 1|H_1(t)|1\rangle = \langle 2|H_1(t)|2\rangle = 0$$

(b) Substitute $|\Psi(t)\rangle = \sum_n c_n(t)|n\rangle e^{-i\frac{E_n t}{\hbar}} = c_1(t)|1\rangle e^{-i\frac{E_1 t}{\hbar}} + c_2(t)|2\rangle e^{-i\frac{E_2 t}{\hbar}}$ into time-dependent Schrodinger equation, $i\hbar\frac{\partial}{\partial t}|\Psi(t)\rangle = H(t)|\Psi(t)\rangle$ where $H(t) = H_0 + H_1(t)$.

Substitute this expression into the Schrödinger equation, and multiplying $\langle 1|$

$$E_1 c_1(t)e^{-i\frac{E_1}{\hbar}t} + i\hbar\dot{c}_1(t)e^{-i\frac{E_1}{\hbar}t} = c_1(t)\langle 1|H_0|1\rangle e^{-i\frac{E_1 t}{\hbar}} + c_2(t)\langle 1|H_1(t)|2\rangle e^{-i\frac{E_2 t}{\hbar}}$$

→

$$\dot{c}_1(t) = \frac{1}{i\hbar}c_2(t)\langle 1|H_1(t)|2\rangle e^{i\frac{E_1-E_2 t}{\hbar}} = \frac{i}{2}\gamma B_1 e^{i(\Omega+\omega)t}c_2(t)$$

(c)

$$\dot{c}_1(t) = \dot{c}_1^{(1)}(t) \approx \frac{i}{2}\gamma B_1 e^{i(\Omega+\omega)t}c_2^{(0)}(t) = \frac{i}{2}\gamma B_1 e^{i(\Omega+\omega)t}$$

$$\rightarrow c_1^{(1)}(t) = \int_0^t \frac{i}{2}\gamma B_1 e^{i(\Omega+\omega)t'}dt' = \frac{i}{2}\gamma\hbar B_1\frac{e^{i(\Omega+\omega)t}-1}{i\hbar(\Omega+\omega)} = \frac{i}{2}\gamma\hbar B_1\frac{\sin\frac{(\Omega+\omega)}{2}t}{i\hbar(\Omega+\omega)}e^{i\frac{(\Omega+\omega)}{2}t}$$

Initially up spin state is not populated, $c_1^{(0)} = 0$.

(d) The transition probability from down to up spin state at t is given by

$$P_{2\to 1}(t) = |c_1^{(2)}(t)|^2 = \frac{\gamma^2\hbar^2 B_1^{\ 2}}{4}\frac{\sin^2\frac{(\Omega+\omega)}{2}t}{\hbar^2(\Omega+\omega)^2}$$

II-7

Solution: II-7

a) We know that, at equilibrium, the chemical potentials of two phases should be equal:

$$\mu_1 [P(\tau), \tau] = \mu_2 [P(\tau), \tau] \tag{1}$$

Here we write $P \equiv P(\tau)$ to emphasize the fact that the pressure depends on the temperature. By taking the derivative of) with respect to temperature, we obtain

$$\left(\frac{\partial \mu_1}{\partial \tau}\right)_P + \left(\frac{\partial \mu_1}{\partial P}\right)_\tau \frac{\mathrm{d}P}{\mathrm{d}\tau} = \left(\frac{\partial \mu_2}{\partial \tau}\right)_P + \left(\frac{\partial \mu_2}{\partial P}\right)_\tau \frac{\mathrm{d}P}{\mathrm{d}\tau} \tag{2}$$

Taking into account that $(\partial \mu / \partial \tau)_P = -s$ and $(\partial \mu / \partial P)_\tau = v$, where s and v are the entropy and volume per particle, and substituting into 2), we have

$$\frac{\mathrm{d}P}{\mathrm{d}\tau} = \frac{s_1 - s_2}{v_1 - v_2} \tag{.3}$$

where subscripts 1 and 2 refer to the two phases. On the other hand, $q = \tau(s_2 - s_1)$, where q is the latent heat per particle, so we can rewrite .3) in the form

$$\frac{\mathrm{d}P}{\mathrm{d}\tau} = \frac{q}{\tau \, \Delta v} \tag{.4}$$

which is the Clausius–Clapeyron equation.

b) Consider the particular case of equilibrium between liquid and vapor. The volume v_l of the liquid is usually much smaller than that for the vapor v_2, so we can disregard v_l in 4) and write

$$\frac{\mathrm{d}P_v}{\mathrm{d}\tau} = \frac{q}{\tau v_v}$$

Using the ideal gas law for vapor, $v_v = \tau / P_v$, we get

$$\frac{\mathrm{d}P_v}{\mathrm{d}\tau} = \frac{q P_v}{\tau^2} \tag{5}$$

or

$$\ln P_v = A - \frac{q}{\tau} \tag{6}$$

We can see that $B = q$. Rewriting .6) in usual units gives

$$\ln P_v = A - \frac{q}{k_B T} = A - \frac{q N_A}{k_B N_A T} = A - \frac{L}{RT}$$

where L is the latent heat per mole, N_A is Avogadro's number, and R is the gas constant.

II-8 [10]

The chemical potential of a single-component gas is given by

$$\mu(T,P) = -RT\ln\left(a\frac{T^{5/2}}{(P+b)}\right),$$

where a, b are material-specific positive constants.

Obtain the internal energy of the system $u(T,v)$.

$$g(T,P) = \mu(T,P)$$

$$v = \left(\frac{\partial g}{\partial P}\right)_T = \frac{RT}{P+b} \quad \Rightarrow \quad (P+b)v = RT$$

(equation of state)

$$g = u - Ts + Pv$$

$$u = g + Ts - Pv \quad , \text{ where } s = -\left(\frac{\partial g}{\partial T}\right)_P =$$

$$= R\ln\left(a\frac{T^{5/2}}{P+b}\right) + \frac{5}{2}R$$

$$u = -RT\ln\left(a\frac{T^{5/2}}{P+b}\right) + RT\ln\left(a\frac{T^{5/2}}{P+b}\right) + \frac{5}{2}RT - (RT - bv)$$

$$= \frac{3}{2}RT + bv$$

$$\boxed{u(T,v) = \frac{3}{2}RT + bv}$$

[II-9] 2. Classical Statistical Mechanics. (20.

$$E = -(n_1 - n_2)\mu H$$

constraint : $n_1 + n_2 = N$ number of particles.

$E = (N - 2n_1)\mu H$ $\Rightarrow N_2 = N - n_1$

a) Given an energy E, there are

$$\frac{N!}{(N-n_1)!\, n_1!}$$

states that give the same energy

$$\Omega(E) = \frac{N!}{(N-n_1)!\, n_1!} \qquad \text{with } n_1 = \frac{N\mu H - E}{2\mu H} = \frac{N}{2} - \frac{E}{2\mu H} \equiv \frac{N}{2} - \frac{E}{2\Delta}$$

$2\Delta \equiv$ Energy gap between spin states.

$$\ln \Omega(E) = \ln N! - \ln((N-n_1)!) - \ln(n_1!)$$

Sterling's approximation $\ln m! = m \ln m - m$ + smaller terms

$$\ln \Omega(E) = N \ln N - N - (N-n_1)\ln(N-n_1) + (N-n_1) - n_1 \ln n_1 + n_1$$

$$= N \ln N - n_1 \ln n_1 - (N-n_1)\ln(N-n_1)$$

$$= -N \left\{ \frac{n_1}{N} \ln\left(\frac{n_1}{N}\right) + \frac{N-n_1}{N} \ln\left(\frac{N-n_1}{N}\right) \right\}$$

$$= -N \left\{ \left(\frac{1}{2} - \frac{E}{2N\Delta}\right) \ln\left(\frac{1}{2} - \frac{E}{2N\Delta}\right) + \left(\frac{1}{2} + \frac{E}{2N\Delta}\right) \ln\left(\frac{1}{2} + \frac{E}{2N\Delta}\right) \right\}$$

$$\boxed{\ln \Omega(E) = -\frac{N}{2} \left\{ \left(1 - \frac{E}{2N\Delta}\right) \ln\left(1 - \frac{E}{2N\Delta}\right) + \left(1 + \frac{E}{2N\Delta}\right) \ln\left(1 + \frac{E}{2N\Delta}\right) - 2\ln 2 \right\}}$$

(2.2

$$S(E) = K_B \ln \Omega(E) \qquad \text{Entropy.}$$

Extensive.

$$S(E) = -\frac{NK_B}{2}\left\{ \left(1 - \frac{E}{2N\Delta}\right)\ln\left(1 - \frac{E}{2N\Delta}\right) + \left(1 + \frac{E}{2N\Delta}\right)\ln\left(1 + \frac{E}{2N\Delta}\right) - 2\ln 2 \right\}$$

b) Absolute temperature is defined as

$$\frac{1}{T} = \left(\frac{\partial S}{\partial E}\right)_N = \frac{1}{2N\Delta}\left(\frac{\partial S}{\partial\left(\frac{E}{2N\Delta}\right)}\right)_N$$

$$-\frac{4\Delta}{K_B T} = \frac{\partial}{\partial x}\left\{ (1 - x)\ln(1-x) + (1+x)\ln(1+x) - 2\ln 2 \right.$$

$$\boxed{-\frac{4\Delta}{K_B T} = \log\left(\frac{1 + \frac{E}{2N\Delta}}{1 - \frac{E}{2N\Delta}}\right)}$$

$$\boxed{\frac{E}{2N\Delta} = -\tanh\left(\frac{2\Delta}{K_B T}\right)}$$

c) under which circumstances is T negative?

If we force the system to a non-equilibrium population where more spins are aligned anti-parallel to the field. This will give a positive energy which cannot be reached if $T \geq 0$.

Under these conditions $\frac{1 + E/2N\Delta}{1 - E/2N\Delta} > 1$, $\ln(\,) > 0$

and $T < 0$; i.e., the temperature is negative.

e) Which is hotter?

If we put a system with negative temperature in contact with a system with positive temperature, an let them reach equilibrium, energy will flow from the higher system with higher energy to the system with lower energy. The system with higher energy is the system with negative temperature. Therefore, the system with negative temperature is hotter!

N.B. The natural variable is not T but $\frac{1}{T}$.

$-\frac{1}{|T|_{neg}} < \frac{1}{T_{pos}}$; the hotter the system, the smaller $\frac{1}{T}$!

II-10 [10]

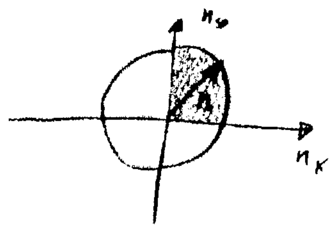
Consider a *two-dimensional photon gas* confined to an area A. What is the average number of photons in the system at temperature T?
Your answer must be expressed in terms of A, T, and of course, the necessary fundamental constants.

To keep your expressions relatively compact, the following relationship will come useful: $\int_0^\infty \frac{x^{\nu-1}}{e^x-1}dx = \Gamma(\nu)\zeta(\nu)$, where $\Gamma(\nu)$ is the gamma function and $\zeta(\nu)$ is the Rieman zeta function. In particular, $\zeta(2) = \frac{\pi^2}{6}$.

$\vec{k} = \frac{\pi}{L}(n_x, n_y) \qquad n_{x,y} = 1,2,3,\ldots$

$$\nu = \frac{\omega}{2\pi} = \frac{kc}{2\pi} = \frac{c}{2\pi}\sqrt{k_x^2+k_y^2} = \frac{c}{2\pi}\left(\frac{\pi}{L}\right)\underbrace{\sqrt{n_x^2+n_y^2}}_{=n} = \frac{c}{2L}n$$

$$n = \frac{2L}{c}\nu$$



number of modes with frequency less than ν: $\mathcal{N}(\nu)$

$$\mathcal{N}(\nu) = 2\times\frac{1}{4}\pi n^2 = \frac{1}{2}\pi\frac{4L^2}{c^2}\nu^2 = \frac{2A\pi}{c^2}\nu^2$$

density of modes: $\boxed{g(\nu) = \frac{\partial\mathcal{N}}{\partial\nu} = \frac{4A\pi}{c^2}\nu}$

$$\langle n_\nu\rangle = \frac{1}{e^{\beta h\nu}-1}$$

(Bosons, chemical potential = 0 quasi particles)

$$N_{ph} = \int_0^\infty d\nu\, g(\nu)\langle n_\nu\rangle = \int_0^\infty d\nu\, g(\nu)\frac{1}{e^{\frac{h\nu}{kT}}-1} = \frac{4A\pi}{c^2}\int_0^\infty\frac{d\nu\,\nu}{e^{\frac{h\nu}{kT}}-1} =$$

$$= \frac{4A\pi}{c^2}\left(\frac{kT}{h}\right)^2\int_0^\infty\frac{dx\,x}{e^x-1} = \frac{4\pi A}{c^2}\left(\frac{kT}{h}\right)^2\overbrace{\Gamma(2)\zeta(2)}^{1\cdot\pi^2/6} =$$

$$= \boxed{\frac{2\pi^3 A k^2}{3c^2h^2}T^2} = \boxed{\text{const.}\, A\cdot T^2}$$