Physics PhD Qualifying Examination
Part I – Wednesday, August 26, 2009

Name:____
(please print)
Identification Number:________

STUDENT: Designate the problem numbers that you are handing in for grading in the appropriate left hand boxes below. Initial the right hand box.
PROCTOR: Check off the right hand boxes corresponding to the problems received from each student. Initial in the right hand box.

	1	
	2	
	3	
	4	
	5	
	6	
	7	
	8	
	9	
	10	

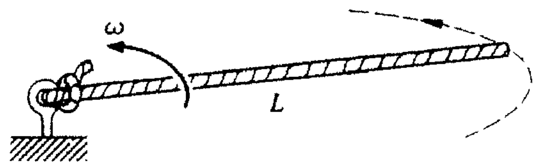
Student's initials
problems handed in:
Proctor's initials

INSTRUCTIONS FOR SUBMITTING ANSWER SHEETS

1. DO NOT PUT YOUR NAME ON ANY ANSWER SHEET. EXAMS WILL BE COLLATED AND GRADED BY THE ID NUMBER ABOVE.
2. Use at least one separate preprinted answer sheet for each problem. Write on only one side of each answer sheet.
3. Write your **identification number** listed above, in the appropriate box on each preprinted answer sheet.
4. Write the **problem number** in the appropriate box of each preprinted answer sheet. If you use more than one page for an answer, then number the answer sheets with both problem number and page (e.g. Problem 9 – Page 1 of 3).
5. Staple together all the pages pertaining to a given problem. Use a paper clip to group together all eight problems that you are handing in.
6. Hand in a total of *eight* problems. A passing distribution will normally include at least three passed problems from problems 1-5 (Mechanics) and three problems from problems 6-10 (Electricity and Magnetism). **DO NOT HAND IN MORE THAN EIGHT PROBLEMS.**
7. **YOU MUST SHOW ALL YOUR WORK.**

[I-1] **[10]**

A uniform rope of mass M and length L is pivoted at one end and whirls with uniform angular velocity ω. What is the tension in the rope at distance r from the pivot? *Neglect gravity.*

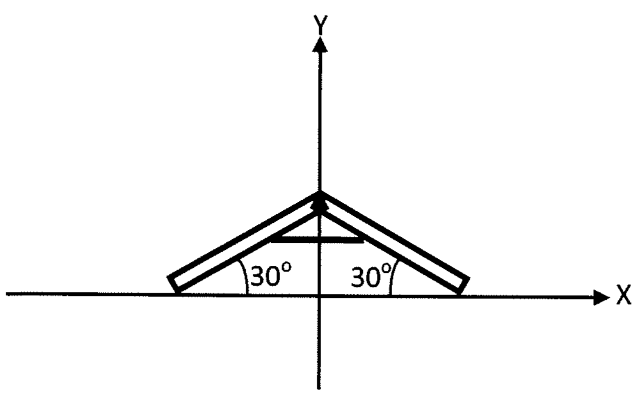


[I-2] **[10]**

Two thin beams of mass m and length l are connected by a frictionless hinge and thread. The system rests on a smooth surface in the way shown in figure below. At $t = 0$ the thread is cut. In the following you may neglect the thread and the mass of the hinge.

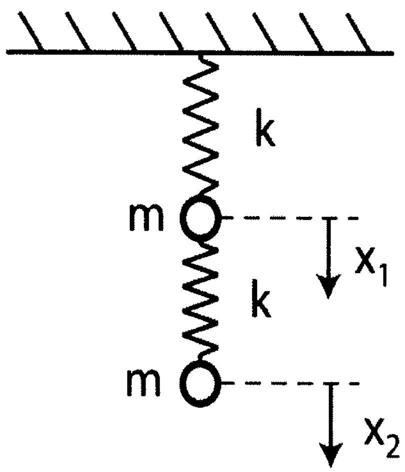
(a) Find the speed of the hinge when it hits the floor.

(b) Find the time it takes for the hinge to hit the floor, expressing this in terms of a concrete integral which you need not evaluate explicitly.



[I-3] **[2,3,3,2]**

Two equal masses are connected vertically with two identical massless springs with spring constant of k and the first spring is hanging from rigid ceiling as shown in the figure. Horizontal broken lines indicate the equilibrium positions of the two masses when they are hanging from the ceiling. x_1, and x_2 are displacement vectors of two masses from their equilibrium positions. The displacements of the masses can be considered as one dimensional motion in the vertical direction. The springs follow Hooke's law. Two masses are under the influence of gravity.



(a) Write down the equations of motion for two masses.
(b) Calculate the eigenfrequencies of normal modes.
(c) Calculate normalized eigenvectors for each of normal modes.
(d) If the whole system is lying on the frictionless horizontal surface and the effect of the gravity along the direction of the motion of the masses can be ignored instead of hanging from the ceiling, what would be the difference of the motions of two masses and what would be the difference in the eigenfrequencies of the system.

[I-4] **[10]**

Find the horizontal deflection from the plumb line caused by the *Coriolis force* acting on a particle falling freely in the Earth's gravitational field from a height h above the Earth's surface. Assume the experiment is conducted in Troy, which has an angle λ relative to the equatorial plane. (I.e., on the equator $\lambda = 0$, whereas at the North Pole $\lambda = \pi/2$.)

[I-5] **[10]**

A qualitative difference between classical mechanics and relativity is the existence of the *transverse Doppler effect* in relativity (when light propagates perpendicular to its source in the observer's frame). Calculate the frequency of the photon ω' in the observer's frame in terms of its frequency ω in the rest frame.

[I-6] [10]

Calculate the potential at any point for a case in which a dielectric sphere of radius a is placed in a uniform field E_o. The dielectric constant outside the sphere is ε_o, the dielectric constant inside the sphere is ε_1.

[I-7] [2,3,3,2]

(a) Write down Maxwell's equations for free space when there are no current nor charge distributions.

(b) Derive wave equations from Maxwell's equation.
Use following relation if necessary. $\nabla\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot(\nabla\times\mathbf{A}) - \mathbf{A}\cdot(\nabla\times\mathbf{B})$

(c) Assume the form of plane wave solution,

$$\mathbf{E}(r,t) = \mathbf{E}_0\cos(\mathbf{k}\cdot\mathbf{r} - \omega t + \phi)$$

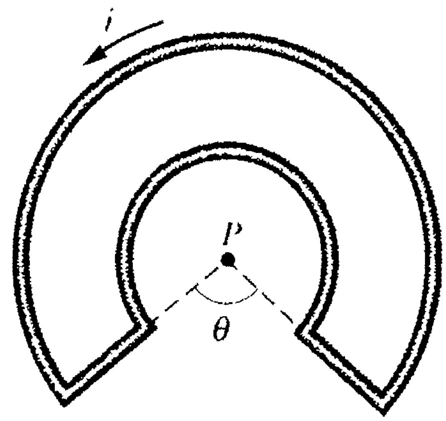
$$\mathbf{B}(r,t) = \mathbf{B}_0\cos(\mathbf{k}\cdot\mathbf{r} - \omega t + \phi)$$

Where $\mathbf{E}_0, \mathbf{B}_0$ are constant in both time and space, **k**,r are wave vector and position vector, ω is angular speed, ϕ is initial phase.
Show that $\mathbf{E}, \mathbf{B}, \mathbf{k}$ are perpendicular to each other so that these vectors form right handed coordinate system.

(d) The magnitude of the pointing vector, $\mathbf{S} = \frac{1}{\mu_0}\mathbf{E}\times\mathbf{H}$, represents the intensity of electromagnetic wave. However for light, it is common to use cycle averaged intensity, $I \equiv |\overline{S}|$ where bar above S indicates cycle average (average over a period of oscillation).
Calculate $I \equiv |\overline{S}|$ for the case of plane wave in vacuum.

[I-8] [10]

A closed loop carries a current i. The loop consists of two radial straight wires and two concentric circular arcs of radii R_1 and R_2. ($R_1 < R_2$) The angle is $\vartheta = \pi/4$. What are the magnitude and direction of the net magnetic field at the center of the curvature, point P?



[I-9] **[3,7]**

(a) Two tiny metal spheres separated by a distance s and connected by a fine wire as shown in the figure below. At time t the charge on the upper sphere is $q(t)$, and the charge on the lower sphere is $-q(t)$. Suppose further that we can drive the charge back and forth through the wire, from one end to other, at frequency ω:

$$q(t) = q_o \cos(\omega t).$$

The system is a simple model of an oscillating electric dipole: $\boldsymbol{p}(t) = p_o \cos(\omega t)\,\hat{z}$, where $p_o = q_0 s$.

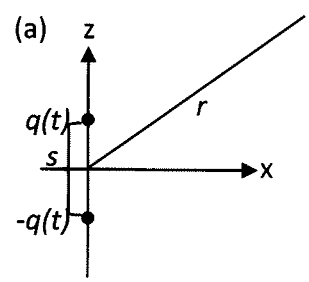
In the far-field approximation $r >> s$, calculate the vector potential of this dipole system and express the vector potential in terms of $\boldsymbol{p}(t)$.

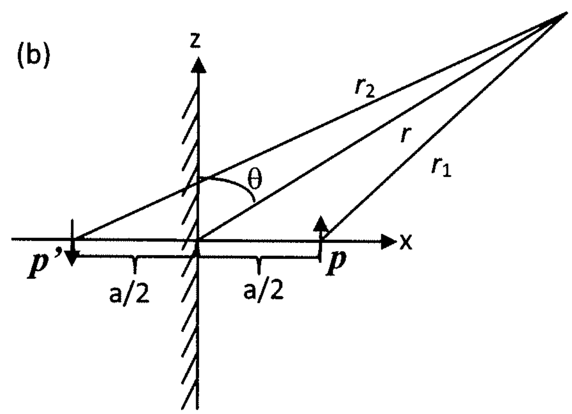
(b) An electric dipole, $\boldsymbol{p}$, oscillates with a frequency ω and amplitude p_o. It is placed at a distant $+a/2$ from an infinite perfectly conducting plane and the dipole is parallel to the plane as shown in the figure below.

Find the time-averaged Poynting vector for distance $r >> a$.

Hint: there is an imaging dipole, $\boldsymbol{p}'$, formed by the conducting plane.)

Note: for approximation: $r_1 \approx r - (a/2)\sin(\vartheta)\cos(\varphi)$; $r_2 \approx r + (a/2)\sin(\vartheta)\cos(\varphi)$; $1/r_1 \approx 1/r_2 \approx 1/r$; also neglect higher order terms of 1/r when you calculate the field.





[I-10]   [10]

Prove that the electric field $\boldsymbol{E}$ and the magnetic induction $\boldsymbol{B}$ of a point charge q with velocity $\boldsymbol{v} = v\hat{\boldsymbol{x}}$ are given by:

$$\boldsymbol{E} = \frac{\gamma q \boldsymbol{R}}{4\pi\varepsilon_o(\gamma^2 R_x^2 + R_y^2 + R_z^2)^{3/2}} , \quad \boldsymbol{B} = \frac{\boldsymbol{v} \times \boldsymbol{E}}{c^2} ,$$

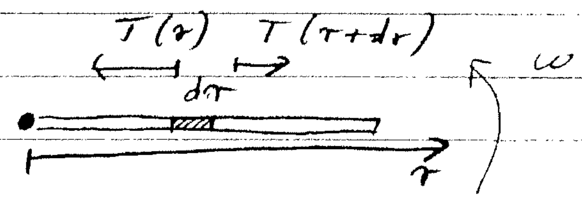
where

$$\boldsymbol{R} = (x - vt)\hat{\boldsymbol{x}} + y\hat{\boldsymbol{y}} + z\hat{\boldsymbol{z}} \quad \text{and} \quad \gamma = \frac{1}{\sqrt{1 - v^2/c^2}} .$$

Solutions

Part I

$\boxed{I-1}$



M, L

equation of motion for a small segment dr (of mass dm):

$$\boxed{T(r+dr) - T(r) = -dm \cdot r \cdot \omega^2}$$

and $$dm = \frac{M}{L}\,dr$$

$$T(r+dr) - T(r) = -\frac{M}{L}\,dr \cdot r \cdot \omega^2$$

$$\frac{T(r+dr) - T(r)}{dr} = -\frac{M}{L} \cdot r\omega^2$$

$$\frac{dT}{dr} = -\frac{M}{L}\, r \cdot \omega^2$$

$$\int dT = -\frac{M}{L}\omega^2 \int r\,dr$$

$$T(r) = -\frac{M}{2L}\omega^2 r^2 + C$$

The tension at the free end must vanish, $T(L) = 0$, hence: $0 = -\frac{M\omega^2 L^2}{2L} + C \implies C = \frac{M\omega^2 L^2}{2L}$

Finally:

$$\boxed{T(r) = \frac{M\omega^2}{2L}\left(L^2 - r^2\right), \quad 0 \leq r \leq L}$$

(I-2)

By symmetry, the hinge will fall vertically. The center-of-mass of the beams:

(i) $X_1 = \frac{1}{2} l \cos\theta \quad , \quad Y_1 = \frac{1}{2} l \sin\theta$
$X_2 = -\frac{1}{2} l \cos\theta \quad , \quad Y_2 = \frac{1}{2} l \sin\theta$

(ii) Their velocity components:
$\dot{x}_1 = -\frac{1}{2} l \dot{\theta} \sin\theta , \quad \dot{y}_1 = \frac{1}{2} l \dot{\theta} \cos\theta$
$\dot{x}_2 = \frac{1}{2} l \dot{\theta} \sin\theta , \quad \dot{y}_2 = \frac{1}{2} l \dot{\theta} \cos\theta$

(iii) Each beam's moment-of-initial: $ml^2/12$

(iv) The Lagrangian of the system:
$$L \equiv T - V$$
$$= \left(\frac{1}{4} m l^2 \dot{\theta}^2 + \frac{1}{12} m l^2 \dot{\theta}^2\right) - mgl \sin\theta$$
$$\therefore L = \frac{1}{3} m l^2 \dot{\theta}^2 - mgl \sin\theta$$

(v) The Lagrange's Equ.:
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$
$$\therefore \ddot{\theta} + \frac{3g}{2l} \cos\theta = 0$$

(1-2)

As $\ddot{\theta} = \frac{1}{2}\frac{d\dot{\theta}^2}{d\theta}$; $\dot{\theta} = 0$, when $\theta = 30^\circ$

$\Rightarrow \quad \dot{\theta}^2 = \frac{3g}{2l}(1 - 2\sin\theta)$

$\therefore \dot{\theta}(\theta = 0^\circ) = -\sqrt{\frac{3g}{2l}}$

$\therefore |V| = |l\dot{\theta}| = \sqrt{\frac{3gl}{2}}$ #

(b) $t = \int_{30^\circ}^{0^\circ} \frac{d\theta}{\dot{\theta}} = \int_{30^\circ}^{0^\circ} \frac{d\theta}{-\sqrt{\frac{3g}{2l}(1 - 2\sin\theta)}}$

$= \sqrt{\frac{2l}{3g}} \int_{0^\circ}^{30^\circ} \frac{d\theta}{\sqrt{1 - 2\sin\theta}}$

#

I (1)

[Answer to the Normal mode problem]

(a)

$m\ddot{x}_1 = -kx_1 - k\left[x_1 - x_2\right]$ (1)

$m\ddot{x}_2 = -k\left[x_2 - x_1\right]$ (2)

(b)

Assume the solution in the form of $x_1 = A\sin(\omega t)$ and $x_2 = B\sin(\omega t)$, and substitute into (1) and (2). These assumed solutions satisfy the rigid boundary condition at ceiling.

From (1), $m\ddot{x}_1 = -kx_1 - k\left[x_1 - x_2\right]$

$$-\omega^2 mA\sin(\omega t) = -kA\sin(\omega t) - k\left[A - B\right]A\sin(\omega t)$$

$$-\omega^2 mA = -kA - k\left[A - B\right]$$

$$-\omega^2 mA + 2kA - kB = 0 \quad (3)$$

From (2), $m\ddot{x}_2 = -k\left[x_2 - x_1\right]$

$$-\omega^2 mB\sin(\omega t) = -k\left[B - A\right]\sin\omega t$$

$$-\omega^2 mB = -k\left[B - A\right]$$

$$-kA + (-\omega^2 m + k)B = 0 \quad (4)$$

These two equations will have the solution if the following determinant is zero.

$$\begin{vmatrix} -\omega^2 m + 2k & -k \\ -k & -\omega^2 m + k \end{vmatrix} = 0$$

$$\rightarrow \left(-\omega^2 m + 2k\right)\left(-\omega^2 m + k\right) - k^2 = 0$$

$$\rightarrow m^2\omega^4 - 3mk\omega^2 + k^2 = 0$$

$$\rightarrow \omega^2 = \frac{3mk \pm \sqrt{9m^2k^2 - 4m^2\left(k^2\right)}}{2m^2} = \frac{3mk \pm \sqrt{5m^2k^2}}{2m^2} = \frac{\left(3 \pm \sqrt{5}\right)k}{2m}$$

Eigen frequencies: $\omega = \sqrt{\dfrac{\left(3 \pm \sqrt{5}\right)k}{2m}}$

(c) Eigen vector:

For higher frequency mode with $\omega = \sqrt{\frac{\left(3+\sqrt{5}\right)k}{2m}}$, substitute this into (4)

$$-kA + (-\omega^2 m + k)B = 0$$

$$-kA + (-\frac{\left(3+\sqrt{5}\right)k}{2m}m + k)B = 0$$

$$A = (-\frac{1+\sqrt{5}}{2})B$$

→Normalization condition→

$$A^2 + B^2 = B^2 + (-\frac{1+\sqrt{5}}{2})^2 B^2$$

$$= \left(1+3+\sqrt{5}\right)B^2 \equiv 1$$

→ $B = \frac{1}{\sqrt{4+\sqrt{5}}}$

→

$$A = (-\frac{1+\sqrt{5}}{2})B = (-\frac{1+\sqrt{5}}{2})\frac{1}{\sqrt{4+\sqrt{5}}} = -\frac{1+\sqrt{5}}{2\sqrt{4+\sqrt{5}}}$$

Eigen vectors: $\begin{pmatrix} A \\ B \end{pmatrix} = \frac{1}{\sqrt{4+\sqrt{5}}}\begin{pmatrix} -\frac{1+\sqrt{5}}{2} \\ 1 \end{pmatrix}$

Similarly, for the lower frequency mode with $\omega = \sqrt{\frac{\left(3-\sqrt{5}\right)k}{2m}}$

$$-kA + (-\omega^2 m + k)B = 0$$

$$-kA + (-\frac{\left(3-\sqrt{5}\right)k}{2m}m + k)B = 0$$

$$A = (-\frac{1-\sqrt{5}}{2})B$$

➔ Normalization condition:

➔ $A^2 + B^2 = (-\frac{1-\sqrt{5}}{2})^2 B^2 + B^2$

$$= \left(4-\sqrt{5}\right)B^2 = 1$$

→ $B = \frac{1}{\sqrt{4-\sqrt{5}}}$

→ Eigen vector:

→ $\begin{pmatrix} A \\ B \end{pmatrix} = \frac{1}{\sqrt{4-\sqrt{5}}} \begin{pmatrix} -\frac{1-\sqrt{5}}{2} \\ 1 \end{pmatrix}$

(d) The effect of the gravity only changes the equilibrium position of masses, and the spring follows Hooke's law. There will be no difference in the eigen value and eigen vector.

I-4

Solution I.4 The acceleration of the particle in the rotating coordinate system fixed on the Earth is

$$\boldsymbol{a}_r = \boldsymbol{g} - 2\boldsymbol{\omega} \times \mathbf{v}_r$$

The acceleration due to gravity $\boldsymbol{g}$ is along the plumb line. We choose the z-axis parallel but opposite to $\boldsymbol{g}$. Let's also pick $\hat{\boldsymbol{x}}$ to be South and $\hat{\boldsymbol{y}}$ to be East. Troy is in the Northern Hemisphere, so

$$\omega_x = -\omega \cos\lambda, \quad \omega_y = 0, \quad \omega_z = \omega \sin\lambda$$

Next, working to leading order in the components of the velocity (in other words, $\mathbf{v} \approx -v\hat{\boldsymbol{z}}$ when computing the Coriolis force), we have

$$\boldsymbol{\omega} \times \mathbf{v}_r = \det \begin{pmatrix} \hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}} \\ -\omega\cos\lambda & 0 & \omega \sin\lambda \\ 0 & 0 & -gt \end{pmatrix} = -\omega g t \cos\lambda \; \hat{\boldsymbol{y}}$$

Then the acceleration components are

$$(\boldsymbol{a}_r)_x = 0, \quad (\boldsymbol{a}_r)_y = 2\omega g t \cos\lambda, \quad (\boldsymbol{a}_r)_z = -g$$

Integration gives us the motion:

$$y = \frac{1}{3}\omega g t^3 \cos\lambda, \quad z = h - \frac{1}{2}gt^2$$

Solving the second equation for the time to fall height h, so that $z = 0$, we immediately find the deflection d in the y direction:

$$d = \frac{1}{3}\omega \cos\lambda \sqrt{8h^3/g}$$

 John Schroeder

[I-5] Solution

Intuitively it is clear that the effect is due to a time dilation $t' = \gamma t$ so
$\omega' = \frac{\omega}{\gamma}$, $(\omega \propto \frac{1}{t})$. More formally we can use the energy and momentum transforma but it is more convenient to introduce a 4-vector k^μ (with $\vec{k}$ a wave vector),

$$k^\mu = \left(\frac{\omega}{c}, \vec{k}\right)$$

and consider its transformation from the rest frame K to the observer's frame K'

$$k^0 = \gamma(k'^0 - \beta k'^1) \quad (\beta = v/c)$$

with $k'^0 = \frac{\omega'}{c}$, $k'^1 = k' \cos\alpha = 0$, $(\alpha = \pi/2)$

Substituting this into: $k^0 = \gamma(k'^1 - \beta k'^1)$

$$k^0 = \frac{\omega}{c} = \gamma \frac{\omega'}{c} \quad \text{and}$$

$$\omega' = \frac{\omega}{\gamma} = \omega\sqrt{1-\beta^2}$$

Note that the transverce Doppler effect gives only a second order correction to the frequency

$$\omega' \approx \omega - \tfrac{1}{2}\omega\beta^2, \quad \text{when } \beta \ll 1,$$

whereas the longitudinal Doppler effect yields a first order correction.

I-6

Solution

Azimutal symmetry

$$\phi(r,\theta) = \sum_{l=0}^{\infty} \left[A_l r^l + B_l r^{-(l+1)} \right] P_l(\cos\theta)$$

Find solutions inside and outside the sphere.
avoid singularities, ∴

$$\Phi_{ext}(r,\theta) = \Phi_0 + \sum \frac{B_l}{r^{l+1}} P_l(\cos\theta)$$

$$\Phi_{in}(r,\theta) = \Phi_0 + \sum A_l r^l P_l(\cos\theta)$$

$$\Phi_0 = -E_0 z = -E_0 r P_1(\cos\theta)$$

Boundary conditions:

(i) $\Phi_{in}(a,\theta) = \Phi_{ext}(a,\theta)$

using orthogonality of $P_l(\cos\theta)$

$$\frac{B_l}{a^{l+1}} = A_l a^l$$

$$\boxed{A_l = \frac{B_l}{a^{2l+1}}} \quad (i)$$

(ii) Normal components of D continuous at $r=a$

$$\varepsilon_0 \left.\frac{\partial \phi_{ext}}{\partial r}\right|_{r=a} = \varepsilon_1 \left.\frac{\partial \phi_{in}}{\partial r}\right|_{r=a}$$

$$\varepsilon_0 \left[E_0 P_1(\cos\theta) + \sum_l (l+1) \frac{B_l}{a^{l+2}} P_l(\cos\theta) \right]$$

$$= \varepsilon_1 \left[E_0 P_1(\cos\theta) - \sum_l l A_l a^{l-1} P_l(\cos\theta) \right]$$

$l=0$; $\quad \boxed{B_0 = 0} \Rightarrow \boxed{A_0 = 0}$

$l=1$; $\quad \varepsilon_0 \left[E_0 + 2\frac{B_1}{a^3} \right] = \varepsilon_1 \left[E_0 - A_1 \right]$

$l>1$; $\quad \varepsilon_0 \left[(l+1) B_l / a^{l+2} \right] = \varepsilon_1 \left[-l A_l a^{l-1} \right]$

Using (i)

$$A_1 = \frac{B_1}{a^3}$$

and for $(\ell = 1)$

$$\varepsilon_0 \left[E_0 + \frac{2B_1}{a^3} \right] = \varepsilon_1 \left[E_0 - \frac{B_1}{a^3} \right]$$

$$(2\varepsilon_0 + \varepsilon_1) \frac{B_1}{a^3} = (\varepsilon_1 - \varepsilon_0) E_0$$

$$\boxed{B_1 = \frac{\varepsilon_1 - \varepsilon_0}{2\varepsilon_0 + \varepsilon_1} a^3 E_0}$$

$$\boxed{A_1 = \frac{\varepsilon_1 - \varepsilon_0}{2\varepsilon_0 + \varepsilon_1} E_0}$$

$\underline{\ell > 1}$; use (i)

$$\varepsilon_0 \left[\frac{(\ell+1) B_\ell}{a^{\ell+2}} \right] = \varepsilon_1 \left[- \frac{\ell B_\ell}{a^{\ell+2}} \right]$$

$$\rightarrow B_\ell = 0 \text{ \& } A_\ell = 0 \quad \forall \ \ell > 1.$$

Solution:

$$\boxed{\begin{aligned} \Phi_{in}(r, \theta) &= -E_0 r \cos\theta + \frac{\varepsilon_1 - \varepsilon_0}{\varepsilon_1 + 2\varepsilon_0} E_0 r \cos\theta \\ \Phi_{ext}(r, \theta) &= -E_0 r \cos\theta + \frac{\varepsilon_1 - \varepsilon_0}{\varepsilon_1 + 2\varepsilon_0} E_0 a^3 \frac{\cos\theta}{r^2} \end{aligned}}$$

(1)

$$div\mathbf{E} = 0 \qquad \text{(a)}$$
$$div\mathbf{B} = 0 \qquad \text{(b)}$$
$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad \text{(c)}$$
$$\nabla \times \mathbf{B} = \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \qquad \text{(d)}$$

in SI unit

(2) Using the second equation:

$$\nabla \times (\nabla \times \mathbf{E}) = -\frac{\partial \nabla \times \mathbf{B}}{\partial t}$$

$$\rightarrow \nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} = -\frac{\partial \nabla \times \mathbf{B}}{\partial t} = -\mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2}$$

$$\rightarrow \nabla^2 \mathbf{E} = \mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2} \qquad \text{(e)}$$

Similarly From 4th equation,

$$\nabla \times (\nabla \times \mathbf{B}) = \nabla(\nabla \times \mathbf{B}) - \nabla^2 \mathbf{B} = \mu_0 \varepsilon_0 \frac{\partial \nabla \times \mathbf{E}}{\partial t} = \mu_0 \varepsilon_0 \left(-\frac{\partial \mathbf{B}}{\partial t} \right)$$

$$\rightarrow \nabla^2 \mathbf{B} = \mu_0 \varepsilon_0 \frac{\partial \mathbf{B}}{\partial t} \qquad \text{(f)}$$

(3) Substitute (e) and (f) into (a) (b) (c) (d),

$$div\mathbf{E} = \mathbf{k} \cdot \mathbf{E} = 0$$
$$div\mathbf{B} = \mathbf{k} \cdot \mathbf{B} = 0$$
$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \rightarrow \mathbf{k} \times \mathbf{E} = \omega \mathbf{B}$$
$$\nabla \times \mathbf{B} = \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \rightarrow \mathbf{k} \times \mathbf{B} = -\frac{\omega}{c^2} \mathbf{E}$$

K and E, k and B are perpendicular, and B is in the place formed by k and E.
B is perpendicular to both of k and E. Cross product of k and E is in the direction of B vector. The relative direction of E, B, k form right hand system.

(4) Poyinting vector in SI unit is $|S| = \frac{1}{\mu_0}|\mathbf{E}\times\mathbf{H}| = |\mathbf{E}||\mathbf{H}|\sin\theta = |\mathbf{E}||\mathbf{H}|$ since $E_0 \perp H_0$

And $\frac{|E|}{|H|} = \sqrt{\mu_0 / \varepsilon_0}$

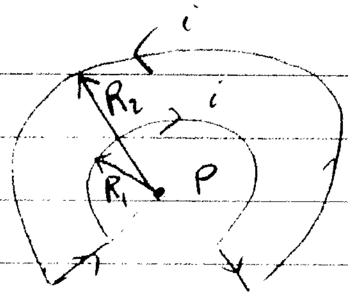
$$\rightarrow |S| = \frac{|\mathbf{E}|^2}{\sqrt{\mu_0/\varepsilon_0}} = \varepsilon_0 c^2 |\mathbf{E}|^2 = \frac{\varepsilon_0 c^2}{2}|\overline{\mathbf{E}}|^2$$

Time average of $|\mathbf{E}|^2$ is given by

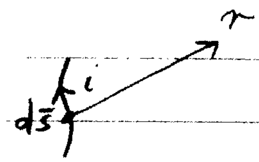
$$|\overline{\mathbf{E}}|^2 = \frac{1}{T}\int_0^T |\mathbf{E}|^2\, dt = \frac{1}{T}\int_0^T \mathbf{E}_0^{\,2}\cos^2(kr - \omega t + \phi)dt$$

$$= \frac{|\mathbf{E}_0|^2}{T}\int_0^T \left(\frac{1}{2} + \cos(2kr - 2\omega t + 2\phi)\right)dt = \frac{|\mathbf{E}_0|^2}{2}$$

$\boxed{\text{I - 8}}$

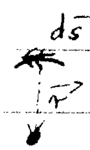


Biot-Savart law:



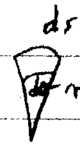
$$d\vec B = \frac{\mu_0}{4\pi}\frac{i\, d\vec s \times \vec r}{r^3}$$

- the two straight segments will have no contribution since they are radial (go through P)
- contribution from a circular arc of radius r (at the center, P



$d\vec s \perp \vec r$ everywhere and $|\vec r| = r$

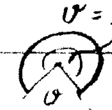
$$dB = \frac{i\mu_0}{4\pi}\frac{ds \cdot r}{r^3} = \frac{i\mu_0}{4\pi}\frac{1}{r^2}\, ds$$



$$ds = r\, d\theta$$

$$dB = \frac{i\mu_0}{4\pi}\frac{1}{r^2}\, r\, d\theta = \frac{i\mu_0}{4\pi}\frac{d\theta}{r}$$

$\tilde\theta = 2\pi - \theta$



for a finite angle $B = \int_0^{\tilde\theta} dB = \int_0^{\tilde\theta} \frac{i\mu_0}{4\pi}\frac{d\theta}{r} = \frac{i\mu_0}{4\pi}\frac{1}{r}\tilde\theta$

$\theta = \pi/4$

Thus the net magnetic field at point P: $\tilde\theta = 2\pi - \theta = 2\pi - \frac{\pi}{4}$

$$B = \frac{i\mu_0}{4\pi}\frac{1}{R_1}\tilde\theta - \frac{i\mu_0}{4\pi}\frac{1}{R_2}\tilde\theta = \frac{i\mu_0\tilde\theta}{4\pi}\left(\frac{1}{R_1} - \frac{1}{R_2}\right) = \frac{i\mu_0 7\pi}{16\pi}\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

$$\boxed{= \frac{7\mu_0 i}{16}\left(\frac{1}{R_1} - \frac{1}{R_2}\right)}$$

direction: <u>into</u> the page

(I-9)

(a) $I(t) = \frac{dq}{dt}\hat{k} = -q_0 \, w \, \sin(wt)\,\hat{k}$

$$\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\int_{-S/2}^{S/2} \frac{-q_0 \, w \sin w(t-\tfrac{r}{c})}{r}\, dz$$

The integration introduces a factor "S".
To first order, we replace the integrad by its value at the center:

$$\vec{A}(\vec{r},t) \cong -\frac{\mu_0 p_0 w}{4\pi r}\sin w(t-\tfrac{r}{c})\,\hat{k}$$

$$= \frac{\mu_0}{4\pi}\frac{\dot{\vec{p}}(t)}{r} \quad \#$$

(b) $\vec{A} = \frac{\mu_0}{4\pi}\left(\frac{\dot{\vec{p}}}{r_1} + \frac{\dot{\vec{p}}'}{r_2}\right)$

$$= -i\frac{\mu_0}{4\pi} w p_0 \left(\frac{e^{ikr_1}}{r_1} - \frac{e^{ikr_2}}{r_2}\right) e^{-iwt}\,\hat{z}$$

(Note: $\vec{p}' = -\vec{p} = -p_0 e^{-iwt}\hat{z}$)

Use the approximation:

$$r_1 \simeq r - \frac{a}{2}\sin\theta\cos\phi$$
$$r_2 \simeq r + \frac{a}{2}\sin\theta\cos\phi$$

and $\frac{1}{r_1} \approx \frac{1}{r_2} \approx \frac{1}{r}$

(I-9)

$$\vec{A} \approx i\frac{\mu_0}{4\pi}\frac{\omega p_0}{r}\left(e^{i\frac{ka}{2}\sin\theta\cos\phi} - e^{-i\frac{ka}{2}\sin\theta\cos\phi}\right)e^{i(kr-\omega t)}\,\hat{z}$$

$$\therefore \vec{A} = -\frac{\mu_0}{2\pi}\frac{\omega p_0}{r}e^{i(kr-\omega t)}\sin\left(\frac{ka}{2}\sin\theta\cos\phi\right)\hat{z}$$

In Spherical coordinates: $\hat{e}_z = \hat{e}_r\cos\theta - \hat{e}_\theta\sin\theta$

$$\vec{B} = \nabla\times\vec{A} \approx \frac{\hat{e}_\phi}{r}\frac{\partial}{\partial r}(rA_\theta)$$

$$= \hat{e}_\phi\frac{i\omega^2 p_0 e^{i(kr-\omega t)}}{2\pi\varepsilon_0 c^3 r}\sin\theta\sin\left(\frac{k}{2}a\sin\theta\cos\phi\right)$$

$$\vec{E} = c\vec{B}\times\hat{e}_r$$

$$\approx \frac{i\omega^2 p_0 e^{i(kr-\omega t)}}{2\pi\varepsilon_0 c^2 r}\sin\theta\sin\left(\frac{k}{2}a\sin\theta\cos\phi\right)\hat{e}_\theta$$

$$\vec{S} = \frac{\varepsilon_0 c}{2}|E|^2\hat{e}_r = \frac{\omega^4 p_0^2\sin^2\theta}{8\pi^2\varepsilon_0 c^3 r^2}\sin^2\left(\frac{k}{2}a\sin\theta\cos\phi\right)\hat{e}_r$$

#

I-10

Solution I.10 In the frame of the charge, it is at rest. Let's temporarily call that the unprimed frame and the lab frame the primed frame. Then:

$$\mathbf{E} = \frac{q\hat{\boldsymbol{r}}}{4\pi\epsilon_0 r^2}, \quad \mathbf{B} = 0. \tag{1}$$

Now just boost to the lab frame. The transformations are:

$$\mathbf{E}'_{\parallel} = \mathbf{E}_{\parallel}, \quad \mathbf{E}'_{\perp} = \gamma(\mathbf{E} + \mathbf{V} \times \mathbf{B})_{\perp}$$

$$\mathbf{B}'_{\parallel} = \mathbf{B}_{\parallel}, \quad \mathbf{B}'_{\perp} = \gamma(\mathbf{B} - \frac{1}{c^2}\mathbf{V} \times \mathbf{E})_{\perp}$$

Because of $\mathbf{B} = 0$, these reduce to

$$\mathbf{E}'_{\parallel} = \mathbf{E}_{\parallel}, \quad \mathbf{E}'_{\perp} = \gamma \mathbf{E}$$

and after taking into account $\mathbf{V} \times \mathbf{E}_{\parallel} = 0$ we can simplify the magnetic induction to

$$\mathbf{B}' = -\frac{\gamma}{c^2}\mathbf{V} \times \mathbf{E}_{\perp} = -\frac{1}{c^2}\mathbf{V} \times \mathbf{E}'_{\perp} = -\frac{1}{c^2}\mathbf{V} \times \mathbf{E}'$$

If the particle is moving with velocity $\mathbf{v} = v\hat{\boldsymbol{x}}$ then the lab frame is moving with opposite velocity $\mathbf{V} = -\mathbf{v}$ relative to the proper frame of the particle. Also the distance from the origin in the rest frame of the particle is

$$r = \sqrt{x^2 + y^2 + z^2}$$

But it sees the lab frame moving in the $-\hat{\boldsymbol{x}}$ direction, so in this relativistic problem

$$x = \gamma(x' - vt') \equiv \gamma R'_x, \quad y = y' \equiv R'_y, \quad z = z' \equiv R'_z$$

For instance, the origin $x' = 0$ of the lab frame is moving with speed v "to the left," and time is dilated $t = \gamma t'$. This means that the distance r appearing in (1) is, in lab frame coordinates

$$r = \sqrt{\gamma^2(x' - vt')^2 + y'^2 + z'^2} \equiv \sqrt{\gamma^2 R'^2_x + R'^2_y + R'^2_z}$$

The unit vector appearing in (1) is just

$$\hat{\boldsymbol{r}} = (x\hat{\boldsymbol{x}} + y\hat{\boldsymbol{y}} + z\hat{\boldsymbol{z}})/r = \frac{\gamma R'_x\hat{\boldsymbol{x}} + R'_y\hat{\boldsymbol{y}} + R'_z\hat{\boldsymbol{z}}}{\sqrt{\gamma^2 R'^2_x + R'^2_y + R'^2_z}}$$

Substituting these results into the above,

$$
\begin{aligned}
E'_x &= \frac{(\gamma R'_x)E_x}{\sqrt{\gamma^2 R'^2_x + R'^2_y + R'^2_z}} = \frac{\gamma R'_x q}{4\pi\epsilon_0(\gamma^2 R'^2_x + R'^2_y + R'^2_z)^{3/2}} \\
E'_y &= \frac{R'_y(\gamma E_y)}{\sqrt{\gamma^2 R'^2_x + R'^2_y + R'^2_z}} = \frac{\gamma R'_y q}{4\pi\epsilon_0(\gamma^2 R'^2_x + R'^2_y + R'^2_z)^{3/2}} \\
E'_z &= \frac{R'_z(\gamma E_z)}{\sqrt{\gamma^2 R'^2_x + R'^2_y + R'^2_z}} = \frac{\gamma R'_z q}{4\pi\epsilon_0(\gamma^2 R'^2_x + R'^2_y + R'^2_z)^{3/2}}
\end{aligned}
$$

and

$$\mathbf{B}' = \frac{1}{c^2}\mathbf{v} \times \mathbf{E}'$$

This proves what was to be shown, once we drop the primes.

Physics PhD Qualifying Examination
Part II – Friday, August 28, 2009

Name:____
(please print)
Identification Number:________

STUDENT: insert a check mark in the left boxes to designate the problem numbers that you are handing in for grading.
PROCTOR: check off the right hand boxes corresponding to the problems received from each student. Initial in the right hand box.

	1	
	2	
	3	
	4	
	5	
	6	
	7	
	8	
	9	
	10	

Student's initials

problems handed in:

Proctor's initials

INSTRUCTIONS FOR SUBMITTING ANSWER SHEETS

1. DO NOT PUT YOUR NAME ON ANY ANSWER SHEET. EXAMS WILL BE COLLATED AND GRADED BY THE ID NUMBER ABOVE.
2. Use at least one separate preprinted answer sheet for each problem. Write on only one side of each answer sheet.
3. Write your **identification number** listed above, in the appropriate box on the preprinted sheets.
4. Write the **problem number** in the appropriate box of each preprinted answer sheet. If you use more than one page for an answer, then number the answer sheets with both problem number and page (e.g. Problem 9 – Page 1 of 3).
5. Staple together all the pages pertaining to a given problem. Use a paper clip to group together all eight problems that you are handing in.
6. Hand in a total of *eight* problems. A passing distribution will normally include at least four passed problems from problems 1-6 (Quantum Physics) and two problems from problems 7-10 (Thermodynamics and Statistical Mechanics). **DO NOT HAND IN MORE THAN EIGHT PROBLEMS.**
7. **YOU MUST SHOW ALL YOUR WORK.**

[II-1]   [10]

.

Compute the leading order *relativistic correction* to the ground state energy of the simple, one-dimensional harmonic oscillator. (Recall the relativistic energy-momentum relationship, $E = \sqrt{p^2c^2 + m^2c^4}$)

[II-2]   [5,5]

One dimensional harmonic oscillator is under weak electric field, and the interaction between the electric field and the oscillator is given by the electric dipole interaction Hamiltonian [Eq.(2)]. Find the energy difference of the oscillator when electric field is turned on. The Hamiltonian of the oscillator is given by $H = H_0 + H'$, where

$$H_0 = \frac{\hat{p}^2}{2m} + \frac{m\omega^2\hat{x}^2}{2} \quad \text{, and} \tag{1}$$

$$H' = -\hat{\mu}\cdot\hat{E} \ . \tag{2}$$

Here $\hat{\mu} = q\hat{x}$, and q is charge associated with the oscillator.

The eigenvalues of the Hamiltonian without the electric field are given by

$E_n = \left(n + \frac{1}{2}\right)\hbar\omega$ where $n = 1, 2, 3, \ldots$

Use the following equation if necessary:

$\langle n|x|n-1\rangle = \langle n-1|x|n\rangle = \sqrt{\frac{n\hbar}{2m\omega}}$.

(a) Assume $H_0 >> H'$, and calculate the lowest order non-vanishing correction to the unperturbed energy.

(b) Draw a sketch of total potential energy when electric field is on and off.

[II-3] **[5,2,2,1]**

Consider an electron spin and an arbitrary unit vector $\vec{e} = (\sin\vartheta\cos\varphi, \sin\vartheta\sin\varphi, \cos\vartheta)$ in the three-dimensional space, specified by the polar (ϑ) and the azimuth (φ) angles. In the usual S_z-representation the electron spin operator can be expressed in terms of the Pauli matrices

$$\vec{S} = \frac{\hbar}{2}\vec{\sigma}, \qquad \vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z),$$

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \; \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \; \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

(a) Determine the properly normalized eigenvectors and eigenvalues (in terms of ϑ and φ) of the following operator:

$$\sigma_e \equiv \vec{e} \cdot \vec{\sigma}.$$

($\vec{e} \cdot \vec{\sigma}$ is the scalar product of the vectors $\vec{e}$ and $\vec{\sigma}$. In units of $\hbar/2$, this is the operator for the spin projected along the direction $\vec{e}$.)

Now assume that we measure S_z, and it is $\hbar/2$.

(b) What is the probability that the component of the spin along the direction $\vec{e}$ is $+\hbar/2$?
(c) What is the probability that the component of the spin along the direction $\vec{e}$ is $-\hbar/2$?
(d) What is the expectation value of the spin along the direction $\vec{e}$?

[II-4] **[10]**

(a) Consider a potential of the form $V(r) = V_o \dfrac{e^{-\alpha r}}{r}$.

(i) Calculate the differential scattering cross section using Born approximation.
(ii) Calculate the total scattering cross section.

(b) Using the results of (a), calculate (*i*) the differential and (*ii*) the total scattering cross section of Coulomb potential $V_C(r) = \dfrac{Z_1 Z_2 e^2}{r}$

[II-5] **[2,2,2,2,2]**

Consider the wavefunction $\psi(r) = Ne^{-r/a_o}$ that describes a particle in three dimensions.

(a) Find N such that $|\psi(r)|^2$ is a probability density.

(b) Determine the expectation values of the moments of the radial position, $\langle r^n \rangle$ for $n = 1, 2$, as well as its variance.

(c) Determine the expectation values of the moments of the momentum, $\langle \hat{p}_r^n \rangle$ for $n = 1, 2$, as well as its variance.

(d) Determine $\langle \Delta \hat{p}_r \rangle \langle \Delta r \rangle$. Does it satisfy the uncertainty principle?

(e) Estimate the kinetic energy of the particle, assuming m is the mass of an electron and a_o is the Bohr radius. Assuming that the particle described by the wave function is an electron in a Hydrogen atom, estimate the potential energy from the kinetic energy and then calculate the total energy. How does it compare to the ground state of a Hydrogen atom?

[II-6] **[4,3,3]**

A hydrogen atom in its *ground state* is placed between the parallel plates of a capacitor. For times $t < 0$, no voltage is applied. Starting at $t = 0$, an electric field $\boldsymbol{E}(t) = E_o \hat{z} e^{-t/\tau}$ is applied, where τ is a constant.

(a) Derive the equation for the probability that the electron ends up in a state j due to this perturbation.

(b) Evaluate the result if the state j is:

(*i*) a 2S state;

(*ii*) a 2P state.

The following expressions may prove useful:

The wavefunction of the 2P state with m=0 is given by:

$$\psi_{210}(r, \vartheta, \varphi) = \frac{1}{\sqrt{32\pi a_o^5}} e^{-r/2a_o} r \cos(\vartheta) \qquad (l = 1, m = 0\ );$$

The wavefunction of the 1S state (ground state) is:

$$\psi_{100}(r, \vartheta, \varphi) = \frac{1}{\sqrt{\pi a_o^3}} e^{-r/a_o} \qquad (l = 0, m = 0\ ),$$

with a_o being the Bohr radius of the hydrogen atom.

[II-7] **[5,5]**

When a particular one component material is in phase α, it obeys an equation of state,

$$\beta P = a + b\beta\mu,$$

where $\beta = 1/T$ and a and b are positive functions of β. Here, P and μ are the pressure and chemical potential, respectively.

When this material is in phase γ,

$$\beta P = c + d(\beta\mu)^2$$

where c and d are positive functions of β, $d > b$, and $c < a$.

(a) Determine the density change that occurs when the material suffers a phase transformation from phase α to phase γ.

(b) What is the pressure at which the transition occurs?

Hint: The Gibbs-Duhem equation, which you may want to use, has the form:

$$0 = SdT - VdP + \sum_{i=1}^{r} n_i d\mu_i ,$$

with S, T, V, and P being the entropy, temperature, volume, and pressure, respectively, while μ_i and n_i are the chemical potential and number of moles of a species i, respectively.

[II-8] **[10]**

Consider the following gas:

$$\left(P + \frac{a}{Tv^2}\right)(v - b) = RT ,$$

where P is the pressure, v is the specific volume, T is the absolute temperature, and a and b are positive material-specific constants. No other information/equations are given about this system, i.e., you can only use the above system-specific information. (Please look carefully; this is *not* the equation of state of the Van der Waals gas)

Determine the volume dependence of the *constant-volume specific heat* $c_v(T,v)$, and write down the most general form of the specific heat of this system, governed and constrained by the laws and mathematical framework of thermodynamics.

[II-9] [3,3,4]

The probability of a particle being in a state "s" of energy E_s in a canonical ensemble is given by

$$P_s = \frac{e^{-E_s/k_BT}}{\sum_r e^{-E_r/k_BT}} = \frac{e^{-E_s/k_BT}}{Z},$$

where $Z = \sum_r e^{-E_r/k_BT}$ is the partition function. Also, $F = -k_BT\ln(Z)$ is the Helmoltz free energy.

(a) Show that the entropy of the system may be expressed as

$$S = -k_B \sum_s P_s \ln(P_s).$$

Using this expression for the entropy, imagine that a system A_1 has a probability $P_r^{(1)}$ of being in a state "r" and a system A_2 has a probability $P_s^{(2)}$ of being in a state "s". Then one can define the entropies S_1 and S_2 like above. Each state of system A consisting of A_1 and A_2 can then be labeled by "r" and "s". Let the probability of A being found in this state is denoted by P_{rs}, and its entropy is defined by

$$S = -k_B \sum_{rs} P_{rs} \ln(P_{rs}).$$

(b) If A_1 and A_2 are weakly interacting (i.e., their states are essentially *independent* of each other), then *show* that the entropy is simply additive, i.e., $S = S_1 + S_2$.
(c) If A_1 and A_2 are *NOT* weakly interacting (i.e., their states are correlated), then *show* that the entropy of the system satisfies $S \le S_1 + S_2$.

Hints: Notice that $P_r^{(1)} = \sum_s P_{rs}$ and $P_s^{(2)} = \sum_r P_{rs}$. Use an appropriate inequality for $\ln(x)$ to simplify the expression for the total entropy.

[II-10]   [10]

A neutrino is an approximately massless particle that travels at the speed of light like a photon, but it has spin 1/2 instead of spin 1. If the neutrinos do not interact with each other ($\mu = 0$), what would be the expression for the distribution of energies $du/d\varepsilon$ ($u = U/V$) in a neutrino gas in an appropriate oven? Assume that the median energy ε_{median} is much greater than the neutrino mass, so that the mass can be neglected.

II - 1

Solutions
Part II

Solution II.1 We have by Taylor series expansion

$$K = E - mc^2 = \sqrt{p^2c^2 + m^2c^4} - mc^2 = \frac{p^2}{2m} - \frac{p^4}{8m^3c^2} + \mathcal{O}(p^6)$$

Thus we want to compute

$$\Delta E = -\frac{1}{8m^3c^2}\langle 0|p^4|0\rangle$$

Now note that

$$p = \frac{\sqrt{m\omega\hbar}}{i\sqrt{2}}(a - a^\dagger)$$

It is a simple algebraic exercise to show that

$$\langle 0|(a - a^\dagger)^4|0\rangle = 3$$

Thus

$$\Delta E = -\frac{3}{32}\frac{(\hbar\omega)^2}{mc^2}$$

II-2

[Answer to time independent perturbation problem]

Since $H_0 >> H'$, the interaction with the electric filed can be treated as perturbation to the system.

The energy of the harmonic oscillator without the electric field is given by

$H_0\psi_n^{(0)} = E_n\psi_n^{(0)}$ and $E_n = \left(n+\frac{1}{2}\right)\hbar\omega$ where $n = 1, 2, 3, \ldots$

For the first order, energy shift is given by the following equation.

$$E_n^{(1)} = \left\langle \varphi_n^{(0)} \middle| H_1 \middle| \varphi_n^{(0)} \right\rangle = -q|E|\left\langle \varphi_n^{(0)} \middle| \hat{x} \middle| \varphi_n^{(0)} \right\rangle = 0$$

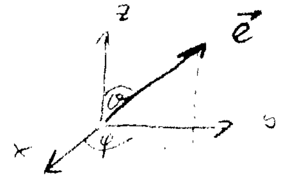
Second order:

$$E_n^{(2)} = \sum_{k\neq n} \frac{\left|\left\langle \varphi_k^{(0)} \middle| H_1 \middle| \varphi_n^{(0)} \right\rangle\right|^2}{E_n^{(0)} - E_k^{(0)}} = q^2|E|^2 \sum_{k\neq n} \frac{\left|\left\langle \varphi_k^{(0)} \middle| \hat{x} \middle| \varphi_n^{(0)} \right\rangle\right|^2}{E_n^{(0)} - E_k^{(0)}}$$

$$= q^2|E|^2 \left[\frac{\left|\left\langle \varphi_{n+1}^{(0)} \middle| \hat{x} \middle| \varphi_n^{(0)} \right\rangle\right|^2}{E_n^{(0)} - E_{n+1}^{(0)}} + \frac{\left|\left\langle \varphi_{n-1}^{(0)} \middle| \hat{x} \middle| \varphi_n^{(0)} \right\rangle\right|^2}{E_n^{(0)} - E_{n-1}^{(0)}} \right]$$

$$= q^2|E|^2 \left[\frac{\left|\sqrt{\frac{(n+1)\hbar}{2m\omega}}\right|^2}{-\hbar\omega} + \frac{\left|\sqrt{\frac{n\hbar}{2m\omega}}\right|^2}{\hbar\omega} \right]$$

$$= q^2|E|^2 \left[-\frac{(n+1)\hbar}{2m\omega^2} + \frac{n\hbar}{2m\omega^2} \right] = -\frac{q^2|E|^2}{2m\omega^2}$$



II.3

a) $\sigma_e = \vec{e}\cdot\vec{\sigma} = \sin\theta\cos\varphi\,\sigma_x + \sin\theta\sin\varphi\,\sigma_y + \cos\theta\,\sigma_z$

$$= \sin\theta\cos\varphi\begin{pmatrix}0&1\\1&0\end{pmatrix} + \sin\theta\sin\varphi\begin{pmatrix}0&-i\\i&0\end{pmatrix} + \cos\theta\begin{pmatrix}1&0\\0&-1\end{pmatrix}$$

$$= \begin{pmatrix}0&\sin\theta\cos\varphi\\\sin\theta\cos\varphi&0\end{pmatrix} + \begin{pmatrix}0&-i\sin\theta\sin\varphi\\i\sin\theta\sin\varphi&0\end{pmatrix} + \begin{pmatrix}\cos\theta&0\\0&-\cos\theta\end{pmatrix}$$

$$= \begin{pmatrix}\cos\theta & \sin\theta\, e^{-i\varphi}\\ \sin\theta\, e^{i\varphi} & -\cos\theta\end{pmatrix}$$

eigenvalues & eigenvectors: $\sigma_e|s_i\rangle = \lambda_i|s_i\rangle \quad i=1,2$

$$\begin{vmatrix}\cos\theta-\lambda & \sin\theta\, e^{-i\varphi}\\ \sin\theta\, e^{i\varphi} & -\cos\theta-\lambda\end{vmatrix} = (\lambda-\cos\theta)(\lambda+\cos\theta)-\sin^2\theta = 0$$

$\lambda^2 - \cos^2\theta - \sin^2\theta = 0$

$\lambda^2 = 1$ $\quad \boxed{\lambda = \pm 1}$ (regardless of the direction $\vec{e}$)

$\underline{\lambda = 1:}$ $(\cos\theta - 1)\,a_1 + \sin\theta\, e^{-i\varphi} a_2 = 0$

$$a_2 = \frac{1-\cos\theta}{\sin\theta} e^{i\varphi} a_1 = \frac{2\sin^2(\theta/2)}{2\sin(\theta/2)\cos(\theta/2)} e^{i\varphi} a_1 = \frac{\sin(\theta/2)}{\cos(\theta/2)} e^{i\varphi} a_1$$

normalized eigenvector:

$$|s_1\rangle = \begin{pmatrix}\cos(\theta/2)\\ \sin(\theta/2)\, e^{i\varphi}\end{pmatrix}$$

$\underline{\lambda = -1:}$ $(\cos\theta + 1)\, a_1 + \sin\theta\, e^{-i\varphi} a_2 = 0$

$$a_2 = -\frac{1+\cos\theta}{\sin\theta} e^{i\varphi} a_1 = -\frac{2\cos^2(\theta/2)}{2\sin(\theta/2)\cos(\theta/2)} e^{i\varphi} a_1 = -\frac{\cos(\theta/2)}{\sin(\theta/2)} e^{i\varphi} a_1$$

normalized eigenvector:

$$|s_2\rangle = \begin{pmatrix} -\sin(\theta/2) \\ \cos(\theta/2)\, e^{i\varphi} \end{pmatrix}$$

b) $|s\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\qquad |s\rangle = c_1 |s_1\rangle + c_2 |s_2\rangle$

$$c_1 = \langle s_1 | s\rangle = (\cos(\theta/2),\ \sin(\theta/2)\, e^{-i\varphi}) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \boxed{\cos(\theta/2)}$$

$$P_1 = |c_1|^2 = \cos^2(\theta/2)$$

c) $$c_2 = \langle s_2 | s\rangle = (-\sin(\theta/2),\ \cos(\theta/2)\, e^{-i\varphi}) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \boxed{-\sin(\theta/2)}$$

$$P_2 = |c_2|^2 = \sin^2(\theta/2)$$

d) $$\langle s | \sigma_e | s\rangle = (\langle s_1 | c_1^* + \langle s_2 | c_2^*)\, \sigma_e\, (c_1 |s_1\rangle + c_2 |s_2\rangle)$$

$$= (\langle s_1 | c_1^* + \langle s_2 | c_2^*)(c_1 |s_1\rangle - c_2 |s_2\rangle) =$$

$$= |c_1|^2 - |c_2|^2 = \cos^2(\theta/2) - \sin^2(\theta/2) = \cos(\theta)$$

$$\langle s | S_e | s\rangle = \langle s | \tfrac{\hbar}{2} \sigma_e | s\rangle = \boxed{\frac{\hbar}{2}\cos(\theta)}$$

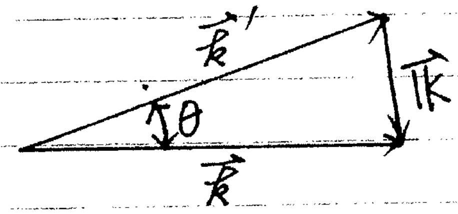
(also simply follows from $\langle \sigma_e \rangle = P_1(+1) + P_2(-1)$)

(II-4)

(a) From Born Approximation:

$$f^{(1)}(\vec{k}',\vec{k}) = -\frac{1}{4\pi}\frac{2\mu}{\hbar^2}\int d\vec{r}^{\,3}\, e^{i(\vec{k}-\vec{k}')\cdot\vec{r}}\, V(\vec{r})$$

(i) $|\vec{k}-\vec{k}'| = 2k\sin\theta/2$



$|\vec{K}| = 2k\sin\theta/2$

(ii) $$f^{(1)} = -\frac{1}{4\pi}\frac{2\mu V_0}{\hbar^2}\int_0^{\infty} r\,dr\ \sin(|\vec{K}|r)\,\frac{e^{-\alpha r}}{r}$$

$$= -\frac{2\mu V_0}{\hbar^2}\frac{1}{\alpha^2+|\vec{K}|^2}$$

$$= \frac{-2\mu V_0}{\hbar^2}\frac{1}{[\alpha^2+4k^2\sin^2\theta/2]}$$

$$\sigma(\theta) = \frac{4\mu^2V_0^2}{\hbar^4}\frac{1}{[\alpha^2+4k^2\sin^2\theta/2]^2}$$

$$\sigma_{total} = \int d\Omega\,\sigma(\theta) = \frac{4\mu^2V_0^2}{\hbar^4}\frac{4\pi}{\alpha^2(\alpha^2+4k^2)}$$

#

③/2

(II-4)

(b) For Coulomb potential, $V = \frac{Z_1 Z_2}{r} e^2$

Set $V_0 = Z_1 Z_2 e^2$ & $\alpha = 0$

(i) $\sigma(\theta) = \frac{4\mu^2}{\hbar^4} \frac{Z_1^2 Z_2^2 e^4}{16 k^4 \sin^4 \theta/2}$

(ii) $\sigma_{total} \to \infty$

#

II-2

Solution:

(a)

$$\int |\psi(r)|^2 d^3r = 1$$

$$4\pi N^2 \int_0^\infty \underbrace{e^{-2r/a_0} r^2 dr}_{a^3/4} = 1$$

$$\boxed{N = \left(\frac{1}{\pi a_0^3}\right)^{1/2}}$$

$$\psi(r) = \left(\frac{1}{\pi a_0^3}\right)^{1/2} e^{-r/a_0}$$

(b)

$$\langle r \rangle = 4\pi \int |\psi|^2 r^3 dr$$

$$= 4\pi \left(\frac{1}{\pi a_0^3}\right) \cdot \int_0^\infty e^{-2r/a_0} r^3 dr$$

$$= 4\pi \cdot \left(\frac{1}{\pi a_0^3}\right) \cdot \frac{3a_0^4}{8} = \frac{3a_0}{2}$$

$$\langle r^2 \rangle = 4\pi \int |\psi|^2 r^4 dr$$

$$= 4\pi \left(\frac{1}{\pi a_0^3}\right) \int_0^\infty e^{-2r/a_0} r^4 dr$$

$$= 4\pi \cdot \left(\frac{1}{\pi a_0^3}\right) \cdot \frac{3a_0^5}{4} = 3a_0^2$$

$$\Delta r^2 = \langle r^2 \rangle - \langle r \rangle^2 = 3a_0^2 - \left(\frac{3}{2}\right)^2 a_0^2 = \frac{3}{4} a_0^2$$

$$\boxed{\Delta r = \left(\frac{3}{4}\right)^{1/2} a_0}$$

(c) $$p_r\psi = -i\hbar \frac{1}{r}\frac{\partial}{\partial r}(r\psi)$$

$$= -i\hbar\left(\frac{\partial}{\partial r} + \frac{1}{r}\right)\psi(r)$$

$$\langle p_r\rangle = \int \psi^*(r)\left(-i\hbar\frac{\partial\psi}{\partial r} - \frac{i\hbar}{r}\psi\right) d^3r$$

$$\frac{\partial\psi}{\partial r} = -\frac{1}{a_0}\psi(r)$$

$$\langle p_r\rangle = \frac{-i\hbar}{a_0} - i\hbar\int\frac{|\psi^*(r)|^2}{r}d^3r$$

$$\int\frac{1}{r}|\psi|^2 d^3r = \frac{4\pi}{\pi a_0^3}\int e^{-2r/a_0}\, r\,dr = \frac{1}{a_0}$$

$$\boxed{\langle p_r\rangle = 0}$$

$$p_r^2 = -\hbar^2\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r}$$

$$\langle p_r^2\rangle = -\hbar^2\int\psi^*(r)\frac{1}{r^2}\left(\frac{\partial}{\partial r}r^2\frac{\partial\psi}{\partial r}\right)d^3r$$

$$= \frac{\hbar^2}{a_0^2}$$

(d) $$\Delta p^2 = \langle p_r^2\rangle - \langle p_r\rangle^2 = \frac{\hbar^2}{a_0^2}$$

$$\Delta p\,\Delta r = \left(\frac{3}{4}\right)^{1/2}\hbar > \hbar$$ uncertainty principle is satisfied

(e) $$K = \frac{\langle p_r^2\rangle}{2m_e} = \frac{\hbar^2}{m_e a_0^2} \quad ; \quad \frac{e^2}{r^2} = \frac{2K}{r} \qquad K = \frac{1}{2}|u|$$

$$E = K - |u| = -\frac{1}{2}K = -\frac{\hbar^2}{2m_e a_0^2} = -\frac{me^4}{2\hbar^2} = -13.60 \text{ eV}.$$

[II-6] Solution

For time-dependent perturbations a general wave function is

$$\Psi(\vec{r},t) = \sum_j a_j(t)\psi_j(\vec{r})e^{-i\omega_j t}$$

with ψ_j satisfying $H_0\psi_j = \hbar\omega_j\psi_j$.
For the time-dependent perturbation $V(t)$

$$V(t) = -e|\vec{E}_0| z\, e^{-t/\tau}.$$

From Schrödinger's equation we can derive an equation for the time ~~derivative~~ development of the amplitudes $a_j(t)$:

$$i\hbar\frac{\partial}{\partial t}\Psi = [H_0 + V(t)]\Psi$$

$$i\hbar\frac{\partial}{\partial t}a_j(t) = \sum_l a_l(t)\langle j|V(t)|l\rangle e^{it(\omega_j-\omega_l}$$

If the system is initially in the ground state, we have $a_{1s}(0) = 1$ and the other values of $a_j(0)$ are zero. For small perturbations it is sufficient to solve the equation for $j \neq 1s$:

$$\frac{\partial}{\partial t}a_j(t) = \frac{ie|\vec{E}_0|}{\hbar}\langle j|z|1s\rangle e^{-t[\frac{1}{\tau}-i(\omega_j-\omega_1}$$

$$a_j(\infty) = \frac{ie|\vec{E}_0|\langle z\rangle}{\hbar}\int_0^\infty dt\, e^{-t[\frac{1}{\tau}-i(\omega_j-\omega_{1s})]}$$

$$a_j(\infty) = \frac{ie|\vec{E}_0|\langle z\rangle\tau}{\hbar[1-i\tau(\omega_j-\omega_{1s})}$$

[II-6] continued.

The general probability P_j that a transition is made to state j is given by

$$P_j = |a_j(\infty)|^2 = \frac{(e|\vec{E}_0|\tau)^2 \langle j|z|1S\rangle^2}{\hbar^2[1+\tau^2(\omega_j-\omega_{1S})^2]}$$

The above probability is dimensionless. It should be less than one for the theory to be valid.

(a.) For the state $j = 2S$ the probability is zero. It vanishes since $\langle 2S|z|1S\rangle = 0$ due to parity. Both S-states have even parity and z has odd parity.

(b.) For the state $j = 2P$ the transition is allowed to $L=1$, $M=0$ orbital state, which is the $2P_z$. The matrix element is similar to the Stark effect problem in hydrogen. The $2P$ eigenstate for $L=1$, $S=0$ is $|10\rangle = \dfrac{z e^{-r/2a_0}}{\sqrt{32\pi a_0^5}}$ and that for the $1S$ state is $e^{(-r/a_0)}/\sqrt{\pi a_0^3}$.

(with: a_0 being the Bohr radius of the hydrogen atom.

The integral becomes

$$\langle 2P_z|z|1S\rangle = \frac{2\pi}{\pi a_0^4\sqrt{32}}\int_0^\infty dr\, r^4 e^{-3r/2a_0}\int_0^\pi d\theta \sin\theta \cos^2\theta$$

$$= \frac{1}{3\sqrt{2}\,a_0^4}\int_0^\infty dr\, r^4 e^{-3r/2a_0} = a_0\left(\frac{2^{3/2}}{3}\right)^5.$$

Ph.D. Q.E. Aug 2009. John Schroeder 7·VI'2009.

[II-7] Solution:

$$\left.\begin{array}{l}\text{phase } \alpha: \beta p = a + b\beta\mu \\ \text{phase } \gamma: \beta p = c + d(\beta\mu)^2\end{array}\right\} \text{ and at phase equilibrium}$$

$\beta p^{(\alpha)} = \beta p^{(\gamma)}$, $\beta\mu^{(\alpha)} = \beta\mu^{(\gamma)}$, and $\beta^{(\alpha)} = \beta^{(\gamma)} = \beta$.

thus $a + \beta b \mu = c + d(\beta\mu)^2$, implies

$$\beta\mu = \frac{b \pm \sqrt{b^2 - 4d(c-a)}}{2d}$$

. Now the density can be obtained from the Gibbs-Duhem equation

$d\mu = -s\,dT + V\,dp$, i.e. $\frac{1}{V} = \rho = \left(\frac{\partial p}{\partial \mu}\right)_T = \left(\frac{\partial(\beta p)}{\partial(\beta\mu)}\right)$

So now: $\rho^{(\alpha)} = b$, $\rho^{(\gamma)} = 2d\beta\mu$.

and we identify the positive root as the physical root to the quadratic equation. Hence,

(i) $\rho^{(\gamma)} - \rho^{(\alpha)} = \sqrt{b^2 + 4d(a-c)}$ and

(ii) $\beta p_{\text{transition}} = a + \frac{b}{2d}\left[b + \sqrt{b^2 + 4d(a-c)}\right]$.

$\boxed{\text{II}-8}$ equation of state: $\left(P + \dfrac{a}{Tv^2}\right)(v-b) = RT$

$$\Rightarrow \quad P = \frac{RT}{v-b} - \frac{a}{Tv^2}$$

$du = Tds - Pdv$

$$\left(\frac{\partial u}{\partial v}\right)_T = T\left(\frac{\partial s}{\partial v}\right)_T - P$$

Maxwell rel:
(from $df = -sdT - Pdv$,
$$\left(\frac{\partial s}{\partial v}\right)_T = \left(\frac{\partial P}{\partial T}\right)_v$$

$$\boxed{\left(\frac{\partial u}{\partial v}\right)_T = T\left(\frac{\partial P}{\partial T}\right)_v - P}$$

for this gas: $\left(\dfrac{\partial u}{\partial v}\right)_T = T\left(\dfrac{R}{v-b} + \dfrac{a}{T^2v^2}\right) - \dfrac{RT}{v-b} + \dfrac{a}{Tv^2} = \dfrac{2a}{Tv^2}$

$$\left(\frac{\partial C_v}{\partial v}\right)_T = \frac{\partial}{\partial v}\left(\frac{\partial u}{\partial T}\right)_v = \frac{\partial^2 u}{\partial v \partial T} = \frac{\partial^2 u}{\partial T \partial v} = \frac{\partial}{\partial T}\left(\frac{\partial u}{\partial v}\right)_T = \text{(from above}$$

$$= \frac{\partial}{\partial T}\left(\frac{2a}{Tv^2}\right) = -\frac{2a}{T^2v^2}$$

$$\Rightarrow \quad C_v = \int \left(\frac{\partial C_v}{\partial v}\right)_T dv + f(T) = \int\left(-\frac{2a}{T^2v^2}\right)dv + f(T)$$

$$\boxed{= \frac{2a}{Tv^2} + f(T)}$$

, where $f(T)$ can only depend on T.

Problem II-9

(a)
$$F = -k_B T \log Z$$

$$S = -\left(\frac{\partial F}{\partial T}\right)_V = k_B \log Z + k_B T \frac{\partial}{\partial T} \log(Z)$$

$$\frac{\partial}{\partial T} \log Z = \frac{1}{T^2} \frac{\sum_s E_s e^{-E_s/k_B T}}{Z} = \frac{1}{T^2} U$$

where $U = \sum E_s \ (s)$.

$$S = k_B \log Z + \frac{1}{T} U = \frac{-k_B}{Z} \sum e^{-E_s/k_B T}\left(-\frac{E_s}{k_B T} - \log Z\right)$$

$$= -k_B \sum P_s \log (P_s^{T})$$

$$S = \frac{U - F}{T}$$, in agreement with the thermodynamic definition of F.

(b)
$$S_1 = -k_B \sum_r P_r^{(1)} \log P_r^{(1)}$$

$$S_2 = -k_B \sum_s P_s^{(2)} \log P_s^{(2)}$$

let $P_{rs} = P_r^{(1)} P_s^{(2)}$

$$S = -k_B \sum_r \sum_s P_{rs} \log P_{rs}$$

$$= -k_B \left\{ \sum_s P_s^{(2)} \sum_r P_r^{(1)} \log P_r^{(1)} + \sum_r P_r^{(2)} \sum P_s^{(1)} \log P_s^{(1)} \right\}$$

$\sum P_r^{(1)} = \sum P_s^{(2)} = 1$

$$S = -k_B \left\{ \sum_r P_r^{(1)} \log P_r^{(1)} + \sum_s P_s^{(2)} \log P_s^{(2)} \right\}$$

$$= S_1 + S_2 .$$

(c)

We cannot assume $P_{rs} = P_r P_s$,

but

$$P_r^{(1)} = \sum_s P_{rs}$$

$$P_s^{(2)} = \sum_r P_{rs}$$

with $\sum_r P_r^{(1)} = \sum_s P_s^{(2)} = 1.$

$$S - (S_1 + S_2) = k_B \sum_{rs} P_{rs} \log\left(\frac{P_r^{(1)} P_s^{(2)}}{P_{rs}} \right)$$

Given that

$$-\log x \geq -x + 1$$

$$S - (S_1 + S_2) \leq k_B \sum_{rs} P_{rs} \left(\frac{P_r^{(1)} P_s^{(2)}}{P_{rs}} - 1 \right)$$

$$\leq 0$$

$\therefore$

$$S \leq (S_1 + S_2).$$

II - 10

Solution II.10

$$dN = \nu V \frac{4\pi p^2 dp}{h^3} n_{\text{Dirac}}(\epsilon(p)) = \nu V \frac{4\pi\epsilon^2 d\epsilon}{(hc)^3(1+e^{\epsilon/k_BT})}$$

where ν is the number of polarizations. Actually, for neutrinos we only have one polarization (left-handed), $\nu = 1$. If antineutrinos (right-handed) are also produced in this oven we get instead $\nu = 2$. In either case,

$$\frac{d(U/V)}{d\epsilon} = \frac{du}{d\epsilon} = \frac{\epsilon d(N/V)}{d\epsilon} = \nu \frac{4\pi\epsilon^3 d\epsilon}{(hc)^3(1+e^{\epsilon/k_BT})}$$