Solutions

Physics PhD Qualifying Examination
Part I – Wednesday, August 20, 2008

Name:____
(please print)
Identification Number:_________

STUDENT: Designate the problem numbers that you are handing in for grading in the appropriate left hand boxes below. Initial the right hand box.
PROCTOR: Check off the right hand boxes corresponding to the problems received from each student. Initial in the right hand box.

	1	
	2	
	3	
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	6	
	7	
	8	
	9	
	10	

Student's initials

\# problems handed in:

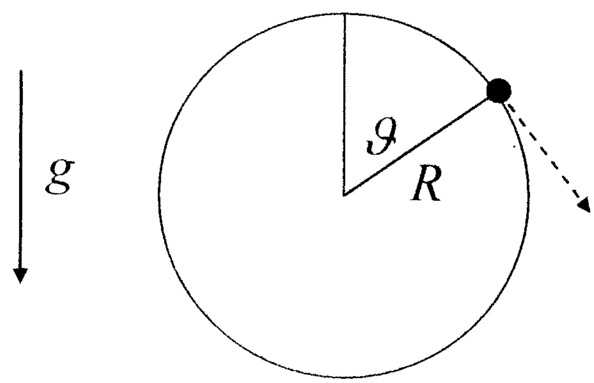
Proctor's initials

INSTRUCTIONS FOR SUBMITTING ANSWER SHEETS

1. DO NOT PUT YOUR NAME ON ANY ANSWER SHEET. EXAMS WILL BE COLLATED AND GRADED BY THE ID NUMBER ABOVE.
2. Use at least one separate preprinted answer sheet for each problem. Write on only one side of each answer sheet.
3. Write your **identification number** listed above, in the appropriate box on each preprinted answer sheet.
4. Write the **problem number** in the appropriate box of each preprinted answer sheet. If you use more than one page for an answer, then number the answer sheets with both problem number and page (e.g. Problem 9 – Page 1 of 3).
5. Staple together all the pages pertaining to a given problem. Use a paper clip to group together all eight problems that you are handing in.
6. Hand in a total of *eight* problems. A passing distribution will normally include at least three passed problems from problems 1-5 (Mechanics) and three problems from problems 6-10 (Electricity and Magnetism). **DO NOT HAND IN MORE THAN EIGHT PROBLEMS.**
7. **YOU MUST SHOW ALL YOUR WORK.**

[I-1] **[10]**

A particle slides freely and without friction on the top surface of a spherically-shaped object with radius R. The mass of the particle is m and the magnitude of the gravitational acceleration is g. The particle is initially at the top of the sphere with infinitesimally small velocity. Determine the angle ϑ at which the particle "takes off", i.e., the angle at which the particle separates from the surface of the sphere. See illustration below.

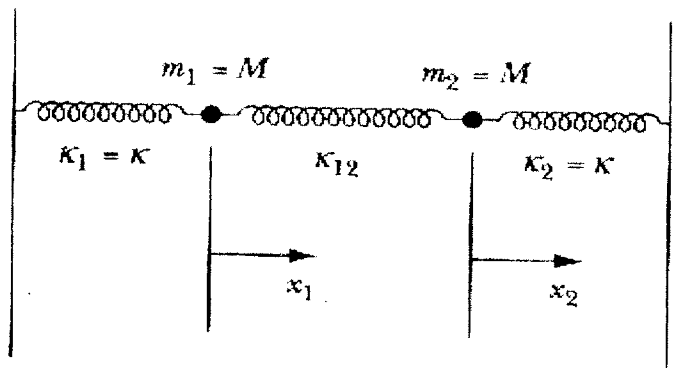


[I-2] **[10]**

A plane pendulum consists of a mass m suspended by a massless spring with unextended length b and spring constant k. Find Lagrange's equations of motion.

[I-3] [10]

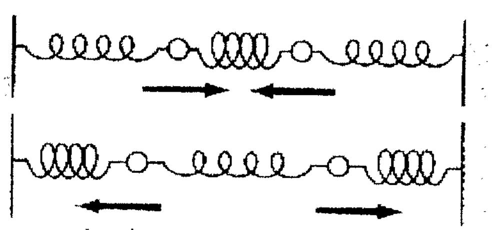
Two masses m_1 and m_2 ($m_1 = m_2 = M$) are connected to each other by a spring with spring constant κ_{12} and to fixed points at the two ends by springs with spring constant κ, as shown below.



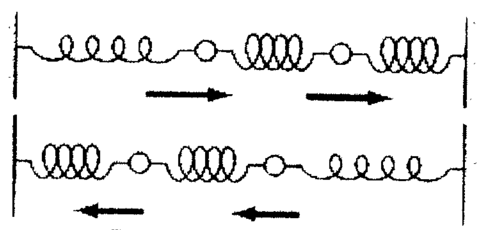
(1) Write down the one-dimensional equations of motion for m_1 and m_2.

(2) Assuming that the motion of the masses is oscillatory (i.e., $x_j(t) = B_j e^{i\omega t}$, $j = 1,2$), simplify the equations of motion; solve this pair of simultaneous equations; and derive the characteristic frequencies (or normal frequencies) for the system.

(3) Given the schematic shown below, associate the appropriate normal modes to the derived characteristic frequencies. Explain your answer (e.g. in terms of either "phase"-argument or "energy"-argument.)



Antisymmetrical mode



Symmetrical mode

[I-4] **[10]**

Two masses m_1 and m_2 are initially separated by a distance r_o and are released from rest. Assume that the only force acting between the two masses is the gravitational force.
Calculate the speed v_1 and v_2 of the two masses as a function of the instantaneous separation r, the initial separation r_o, m_1, m_2, and G (gravitational force constant).

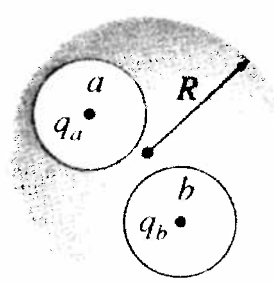
[I-5] **[10]**

An atom in its ground state has mass m. It is initially at rest, in an excited state of excitation energy $\Delta\varepsilon$. It then makes a transition to the ground state by emitting one photon. Find the frequency of the photon, taking into account the *relativistic recoil* of the atom. Express your answer also in terms of the mass M of the excited atom.

[I-6] **[3,3,2,2]**

Two spherical cavities of radii a and b are hollowed out from the interior of a (neutral) conducting sphere of radius R. At the center of each cavity a point charge is placed – call these charges q_a and q_b (see illustration below).

(a) Find the surface charges σ_a, σ_b and σ_R.
[σ_a is the surface charge on the surface of cavity a, σ_b is the surface charge on the surface of cavity b, and σ_R is the surface charge on the conducting sphere.]
(b) What is the electric field outside the conducting sphere?
(c) What is the electric field inside each cavity?
(d) What is the force on q_a and q_b?



[I-7] **[10]**

(1) Write down

(i) the expression for the electric field $\boldsymbol{E}$ in terms of the scalar potential V and the vector potential $\boldsymbol{A}$;

(ii) the relation between the magnetic field $\boldsymbol{B}$ and the vector potential $\boldsymbol{A}$.

(2) Given the scalar potential $V = 0$ and vector potential $\boldsymbol{A}$ below,

$$\boldsymbol{A} = \begin{cases} \dfrac{\mu_o \alpha}{4c}(ct - |x|)^2 \hat{z} & \text{for} \quad |x| < ct \\ 0 & \text{for} \quad |x| > ct \end{cases},$$

find $\boldsymbol{E}$ and $\boldsymbol{B}$.

(3) Write down

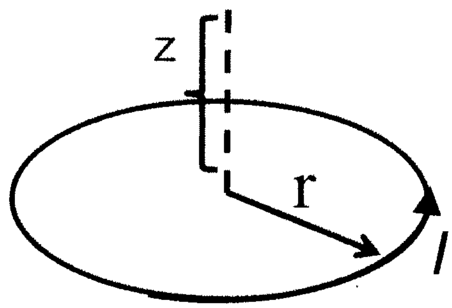
(i) the expression that relates the discontinuity in the electric displacement vector $\boldsymbol{D}$ to the free surface charge density σ_f at the $x = 0$ interface.

(ii) the expression that relates the discontinuity in magnetic field $\boldsymbol{B}$ to the free surface current density $\boldsymbol{K}_f$ at the $x = 0$ interface.

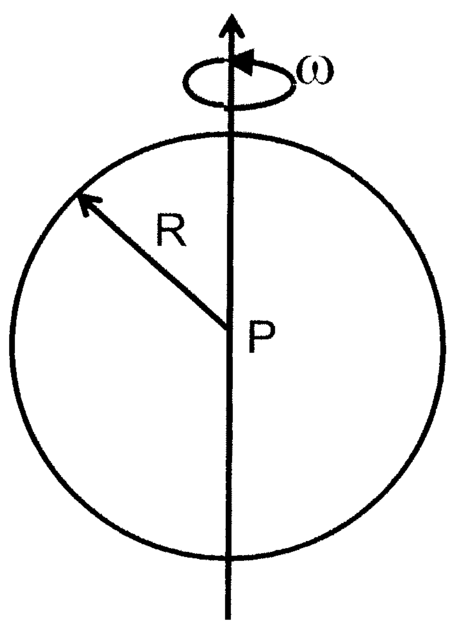
(4) Find the free surface charge and free current density due to the potentials given in (2). (Assume that the dielectric constants and permeabilities on both sides of interface are ε_o and μ_o, respectively).

[I-8] **[10]**

(a) Find the magnetic field at a distance z above the center of a circular loop of radius r, which carries a steady current I, as shown below.

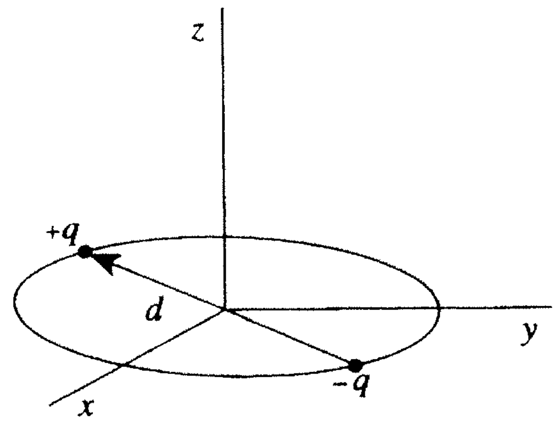


(b) A spherical shell of radius R, carrying a uniform surface charge σ, is set spinning at an angular velocity ω, as shown below. Find the magnetic field at center P of this spherical shell.



[I-9] **[3,4,3]**

Charges $+q$ and $-q$ a distance d apart orbit around each other in the x-y plane ($z = 0$), as shown below, at a frequency ω ($d << c/\omega$).



(a) The emitted radiation is primarily confined to one multipole. Which one?
(b) What is the angular distribution of the radiated power?
(c) What is the total power radiated?

[I-10] **[10]**

The Lorentz transformation of the electromagnetic fields can be written

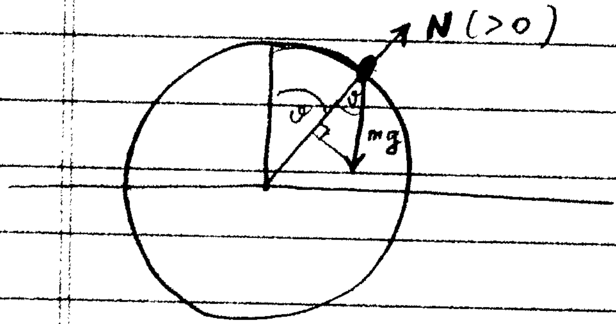
$$\boldsymbol{E}'_{\parallel} = \boldsymbol{E}_{\parallel}, \qquad \boldsymbol{B}'_{\parallel} = \boldsymbol{B}_{\parallel}$$

$$\boldsymbol{E}'_{\perp} = \gamma(\boldsymbol{E}_{\perp} + \boldsymbol{v} \times \boldsymbol{B}), \qquad \boldsymbol{B}'_{\perp} = \gamma(\boldsymbol{B}_{\perp} - \frac{1}{c^2}\boldsymbol{v} \times \boldsymbol{E}_{\perp}) \ ,$$

Where $\parallel$ indicates the component parallel to the velocity $\boldsymbol{v}$ and $\perp$ the perpendicular part. For instance, $\boldsymbol{E}_{\perp} = \boldsymbol{E} - \boldsymbol{E}_{\parallel}$. As usual, $\gamma = (1 - v^2/c^2)^{-1/2}$, where $v = |\boldsymbol{v}|$ is the speed.

(a) Show that the quantities $\boldsymbol{E} \cdot \boldsymbol{B}$ and $\boldsymbol{E}^2 - c^2\boldsymbol{B}^2$ are Lorentz invariants.
(b) Evaluate those quantities for a plane wave in vacuum..

I-1



radial direction: $\dfrac{mv^2}{R} = mg\cos\theta - N$

conservation of energy: $mgR = mgR\cos\theta + \frac{1}{2}mv^2$ $\quad (v_0 = 0)$

$$v^2 = 2gR\,(1-\cos\theta)$$

$$N = mg\cos\theta - m\frac{v^2}{R} = mg\cos\theta - \frac{m}{\cancel{R}}2g\cancel{R}\,(1-\cos\theta) =$$

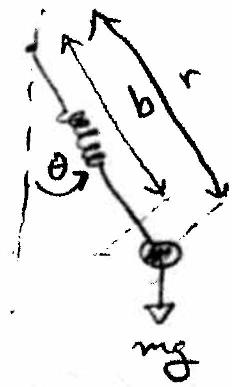
$$= mg\,(3\cos\theta - 2) > 0$$

(N being the normal force has to be positive. It vanishes precisely at the point where the particle "takes off".

Thus, $\quad 3\cos\theta - 2 > 0$

$$\boxed{\cos\theta_c = 2/3} \quad \Rightarrow \theta_c = \cos^{-1}(2/3) \simeq 48.2^\circ$$

Marion & Thornton 7-15



$$T = \frac{1}{2} m r^2 \dot{\theta}^2 + \frac{1}{2} m \dot{r}^2$$

$$V = -mgr\cos\theta + \frac{1}{2} k (r-b)^2$$

$$\frac{\partial L}{\partial \theta} = -mgr\sin\theta$$

$$\frac{\partial L}{\partial r} = mr\dot{\theta}^2 + mg\cos\theta - k(r-b)$$

$$\frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}, \quad \frac{\partial L}{\partial \dot{r}} = m\dot{r}$$

E.O.M. :

$$\boxed{\begin{aligned} 0 &= m\ddot{r} - mr\dot{\theta}^2 - mg\cos\theta + k(r-b) \\ 0 &= mr^2\ddot{\theta} + 2mr\dot{r}\dot{\theta} + mgr\sin\theta \end{aligned}}$$

I-3

(1) When masses m_1 and m_2 are displaced from their equilibrium positions by an amount x_1 and x_2, respectively,

The equ. of motion are:

$$\begin{cases} M\ddot{x}_1 + (k+k_{12})x_1 - k_{12}x_2 = 0 \\ M\ddot{x}_2 + (k+k_{12})x_2 - k_{12}x_1 = 0 \end{cases} \quad \text{------ (1}$$

(2) We assume that the motion is oscillatory and attempt the following solution:

$$\begin{cases} x_1(t) = B_1 e^{i\omega t} \\ x_2(t) = B_2 e^{i\omega t} \end{cases} \quad \text{------ (2)}$$

Combining equ. (1) and (2), we obtain:

$$\begin{cases} (k+k_{12}-M\omega^2)B_1 - k_{12}B_2 = 0 \\ \quad -k_{12}B_1 + (k+k_{12}-M\omega^2)B_2 = 0 \end{cases} \quad \text{---- (3}$$

For non-trivial solutions we demand,

$$\begin{vmatrix} k+k_{12}-M\omega^2 & -k_{12} \\ -k_{12} & k+k_{12}-M\omega^2 \end{vmatrix} = 0 \quad \text{------ (4}$$

I-3

$$\therefore\ \omega_1 = \sqrt{\frac{(k+k_{12})}{M}}^{\;2}$$

$$\omega_2 = \sqrt{\frac{k}{M}}$$

(3) For the symmetrical mode shown, the two masses are always moving "in phase", the spring "k_{12}" is never involved in contributing to the potential term of the motion.

The associated frequency is: $\omega_1 = \sqrt{k/M}$.

For the anti-symmetrica mode, the two masses move "out-of-phase". ~~And~~, the mode energy is higher.

$$\omega_2 = \sqrt{\frac{(k+k_{12})}{M}}^{\;2}$$

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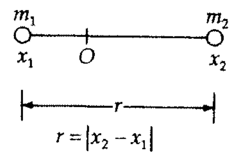
Suggested Solutions

Part I Mechanics
I-4 Gravity or other Central Potential



When two particles are initially at rest separated by a distance r_0, the system has the total energy

$$E_0 = -G\frac{m_1 m_2}{r_0} \tag{1}$$

The coordinates of the particles, x_1 and x_2, are measured from the position of the center of mass. At any time the total energy is

$$E = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 - G\frac{m_1 m_2}{r} \tag{2}$$

and the linear momentum, at any time, is

$$p = m_1\dot{x}_1 + m_2\dot{x}_2 = 0 \tag{3}$$

From the conservation of energy we have $E = E_0$, or

$$-G\frac{m_1 m_2}{r_0} = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 - G\frac{m_1 m_2}{r} \tag{4}$$

Using (3) in (4), we find

$$\boxed{\begin{aligned} \dot{x}_1 = v_1 &= m_2\sqrt{\frac{2G}{M}\left[\frac{1}{r} - \frac{1}{r_0}\right]} \\ \dot{x}_1 = v_2 &= -m_1\sqrt{\frac{2G}{M}\left[\frac{1}{r} - \frac{1}{r_0}\right]} \end{aligned}} \tag{5}$$

[I-5] Solution

Write the energy and momentum conservation equations:

$$p + p_{ph} = 0$$

$$mc^2 + \Delta\mathcal{E} = \hbar\omega + \sqrt{p^2c^2 + m^2c^4} \quad ,$$

where p is the momentum of the atom afte emitting the photon, $p_{ph} = \hbar\omega/c$, the momentu of the photon, and ω is the photon frequen

Substituting $p = -\hbar\omega/c$ from the moment conservation equation into the energy conservation equation above, and rewriting it into the form

$$mc^2 + \Delta\mathcal{E} - \hbar\omega = \sqrt{\hbar^2\omega^2 + m^2c^4} \, ,$$

we find after squaring both sides,

$$\omega = \frac{\Delta\mathcal{E}}{2\hbar}\left(\frac{\Delta\mathcal{E} + 2mc^2}{\Delta\mathcal{E} + mc^2}\right),$$

and taking into account that

$$\Delta\mathcal{E} + mc^2 = Mc^2$$

we may rewrite the above as

$$\omega = \frac{\Delta\mathcal{E}}{\hbar}\left(1 - \frac{\Delta\mathcal{E}}{2Mc^2}\right),$$

which is smaller by the amount of $(\Delta\mathcal{E})^2/2Mc^2\hbar$ than it would have been without the relativistic effect. In the case of a crystalline lattice (Mössbauer effect), the atoms are strongly coupled to the lattice and have an effective mass $M_0 \gg M$. From the equation above we can see that in this case the atom practically does not absorb energy, which all goes into the energy of the photon, and therefore there is no frequency shift due to this effect

Electricity and Magnetism
I-6 Electrostatics or Boundary Value

(a) $\boxed{\sigma_a = -\dfrac{q_a}{4\pi a^2};}$ $\boxed{\sigma_b = -\dfrac{q_b}{4\pi b^2};}$ $\boxed{\sigma_R = \dfrac{q_a + q_b}{4\pi R^2}.}$

(b) $\boxed{\mathbf{E}_{\text{out}} = \dfrac{1}{4\pi\epsilon_0}\dfrac{q_a + q_b}{r^2}\,\hat{\mathbf{r}},}$ where $\mathbf{r}$ = vector from center of large sphere.

(c) $\boxed{\mathbf{E}_a = \dfrac{1}{4\pi\epsilon_0}\dfrac{q_a}{r_a^2}\,\hat{\mathbf{r}}_a, \quad \mathbf{E}_b = \dfrac{1}{4\pi\epsilon_0}\dfrac{q_b}{r_b^2}\,\hat{\mathbf{r}}_b,}$ where $\mathbf{r}_a$ $(\mathbf{r}_b)$ is the vector from center of cavity a (b).

(d) $\boxed{\text{Zero.}}$

I-7

(1) (i) $\vec{E} = -\nabla V - \frac{\partial \vec{A}}{\partial t}$ .... (1)

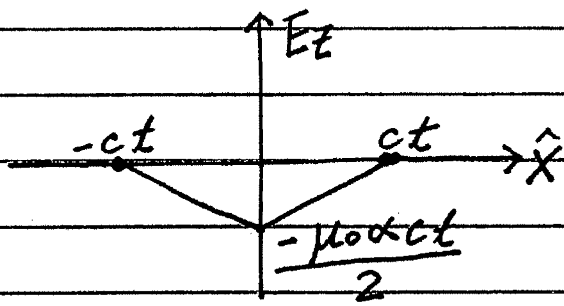
(ii) $\vec{B} = \nabla \times \vec{A}$ .... (2)

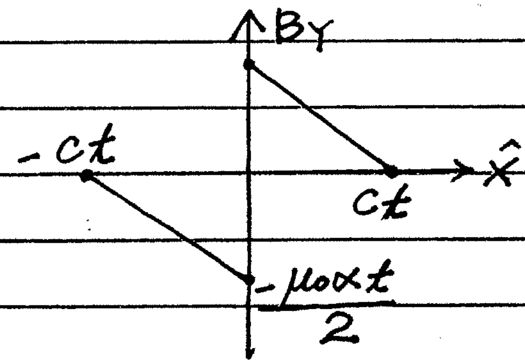
(2) from equ.-(1), $\vec{E} = -\nabla V - \frac{\partial \vec{A}}{\partial t} = -\frac{\mu_0 \alpha}{2}(ct - |x|)\hat{z}$ .... (3)

from equ.-(2), $\vec{B} = \nabla \times \vec{A} = \pm \frac{\mu_0 \alpha}{2c}(ct - |x|)\hat{y}$ ..... (4)

(plus for $x > 0$, minus for $x > 0$.)

(These are for $|x| < ct$; when $|x| > ct$, $\vec{E} = 0 = \vec{B}$





(3) (i) The discontinuity in $\vec{D}$ and σ_f is related:

$D_{1\perp} - D_{2\perp} = \sigma_f$ ———— (5)

("$\perp$" is the component that is perpendicular to the interface.

(ii) The discontinuity in $\vec{B}$ and $\vec{K}_f$ is related:

$\frac{1}{\mu_1}(B_1)_{\parallel} - \frac{1}{\mu_2}(B_2)_{\parallel} = \vec{K}_f \times \hat{n}$ ——— (6)

("$\parallel$" is the component of $\vec{B}$ that is parallel to the interf

("$\hat{n}$" is the interface normal)

I-7

(4) from equ.-(4), $\vec{B}$ has a discontinuity at $x=0$

from equ.-(6), we find:

$$\left(\frac{\alpha t}{2} - \frac{-\alpha t}{2}\right)\hat{y} = \vec{K}_f \times \hat{x}$$

$$\therefore \quad \vec{K}_f = (\alpha t)\,\hat{z}.$$

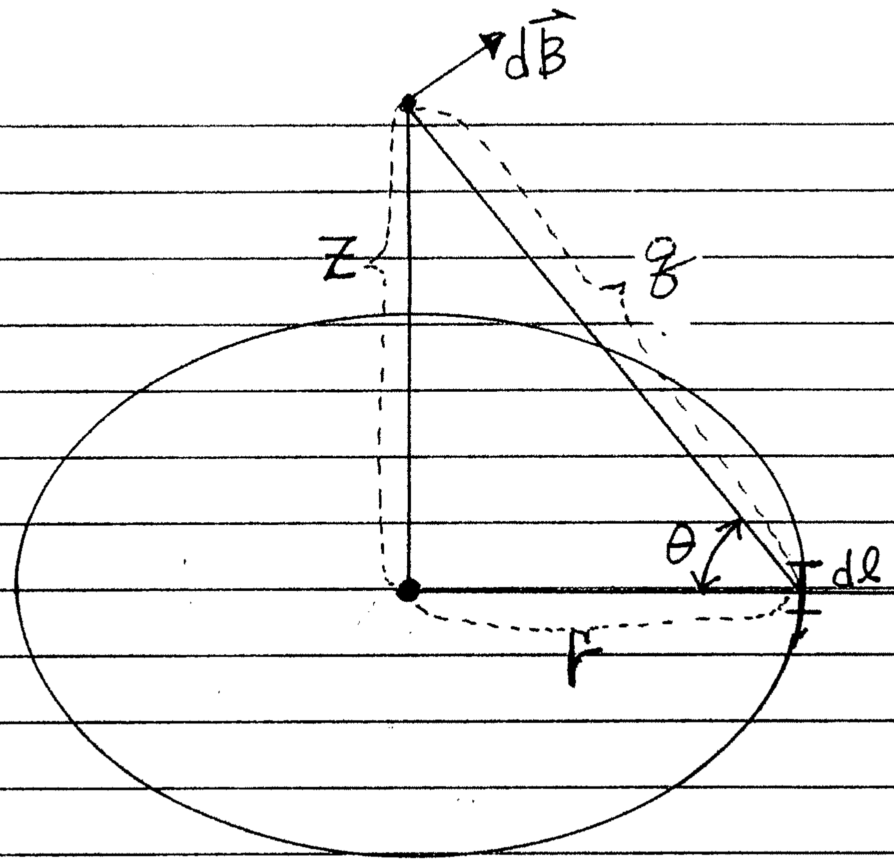
from equ.-(3), $\vec{E}$ is continuous.

There is no free surface charge.

$$\therefore \quad \sigma_f = 0 \quad \#$$

I-8

(1)



By symmetry, the horizontal (or in-plane) component of $\vec{dB}$ produced by the segment dl will be cancelled out as we integrate it over the entire current loop. $\therefore \vec{B}_{\parallel} = 0$

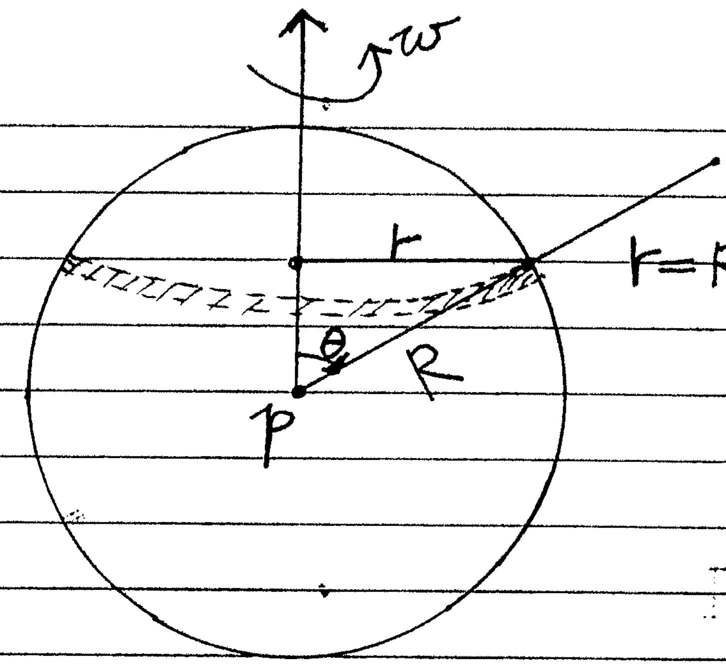
The vertical (or) Z-component of $\vec{dB}$:

$$B = \frac{\mu_0 I}{4\pi} \int \frac{dl \cos\theta}{ᵹ^2}$$

$$B = \frac{\mu_0 I}{4\pi} \frac{2\pi r \cos\theta}{ᵹ^2} = \boxed{\frac{\mu_0 I}{2} \frac{r^2}{(r^2+z^2)^{3/2}} \ldots}$$

II-8

(2)



$r = R\,\cancel{\cos\theta}\,\sin\theta$ ...... (2)

Decompose the rotating spherical shell into a sequence of rotating rings, having a current component dI:

(i) $dI = (\sigma \cdot v)\, R\, d\theta$ .... (3)

Here v is the velocity, $v = \omega r$ ..... (4)

Combining (3) & (4), $\boxed{dI = \sigma \omega r R\, d\theta \;\cdots\cdots (5)}$

(ii) From equ.-(1), the $\vec{B}$-field at point-P is:

$$B = \int_0^{\pi} dB = \frac{\mu_0}{2} \int_0^{\pi} dI \cdot \frac{r^2}{R^3}$$

From equ.-(5), $B = \dfrac{\mu_0}{2} \displaystyle\int_0^{\pi} \frac{(\sigma \omega r R\, d\theta)\, r^2}{R^3}$

From equ.-(2), $B = \mu_0 \displaystyle\int_0^{\pi/2} \frac{\sigma \omega R^4 \sin^3\theta\, d\theta}{R^3}$

$$B = \mu_0 \sigma \omega R \int_0^{\pi/2} \sin^3\theta\, d\theta$$

$$\boxed{\therefore B = \frac{2}{3} \mu_0 \sigma \omega R}$$

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John Schroed

[I-9] Solutions

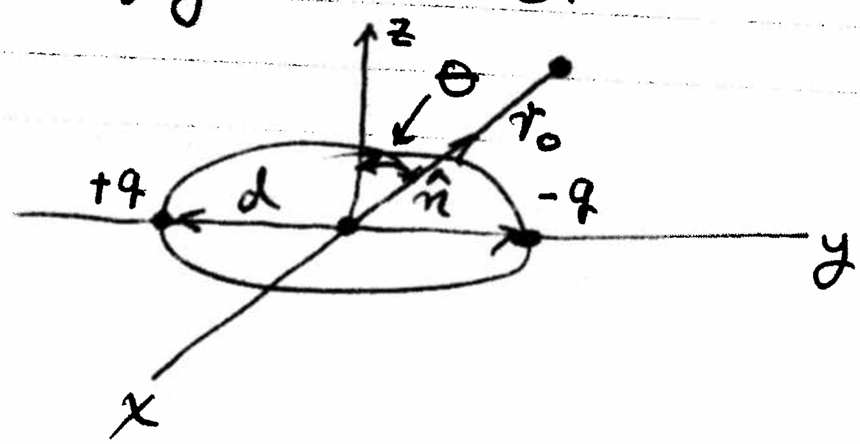
(a, b) At $r \gg d$, the emitted radiation is confined to a dipole ($r \gg \lambda \gg d$) where λ is the wavelength. The vector potential of the system with dipole moment $\vec{p}$ at a distance $r \gg \lambda$ is given by

$$\vec{A} = \frac{1}{c r_0} \dot{\vec{p}} \, .$$

The magnetic field of the system is given by

$$\vec{H} = \frac{1}{c}[\dot{\vec{A}} \times \hat{n}] = \frac{1}{c^2 r_0} \ddot{\vec{p}} \times \hat{n} \, ,$$

where $\vec{p} = q\vec{d}$ is the dipole moment of the system, $\hat{n}$ is the unit vector in the direction of observation, and r_0 is the distance from the origin as shown in the figure below:



The energy flux is given by the Poyting vector "$\vec{S}$":

$$\vec{S} = c \frac{\vec{H}^2}{4\pi} \hat{n}$$

[I-9] solutions - continued

The radiated power in a solid angle $d\Theta$ is given by

$$dP = S r_0^2 \, d\Theta = c \frac{H^2 r_0^2}{4\pi} d\Theta$$

and upon substitution we find

$$dP = \frac{1}{4\pi c^3} |\ddot{\vec{p}} \times \hat{n}|^2 d\Theta$$

Noting that $\vec{p} = q\vec{d} = (p\cos\omega t, p\sin\omega t) = (p_x, p_y)$ we have

$$\ddot{\vec{p}} \times \hat{n} = -\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ p_x & p_y & 0 \\ n_x & n_y & n_z \end{vmatrix} \omega^2 = \Big[-p_y n_z \hat{i} + p_x n_z \hat{j} - (p_x n_y - p_y n_x)\hat{k} \Big] \omega^2$$

$$\langle |\ddot{\vec{p}} \times \hat{n}|^2 \rangle = \langle p^2 \cos^2\theta + p^2 \cos^2\omega t \sin^2\theta \sin^2\varphi + p^2 \sin^2\omega t \sin^2\theta \cos^2\varphi - 2p^2 \sin\omega t \cos\omega t \sin^2\theta \, * \, * \sin\varphi\cos\varphi \rangle \omega^4.$$

Taking the average over the period of revolution:

$\langle \sin^2\omega t \rangle = \langle \cos^2\omega t \rangle = \frac{1}{2}$ and $\langle 2\sin\omega t \cos\omega t \rangle = \langle \sin 2\omega t \rangle = 0$ we find

$$\langle |\ddot{\vec{p}} \times \hat{n}| \rangle = \frac{1}{2} p^2 (1 + \cos^2\theta) \omega^4$$

$$\therefore \langle dP \rangle = \frac{1}{4\pi c^3} \frac{1}{2} p^2 \omega^4 (1 + \cos^2\theta) \, d\Theta$$

[I-9] solution-continued

$\therefore$ (b) $\langle dP \rangle = \frac{p^2 \omega^4}{8\pi c^3}(1+\cos^2\theta)\, d\Omega$

(c) The total power radiated is

$$\langle P \rangle = \int \frac{dP}{d\Omega}\, d\Omega = \frac{p^2\omega^4}{8\pi c^3}\, 2\pi \int_0^{\pi} (1+\cos^2\theta)\sin\theta\, d\theta$$

$$\langle P \rangle = \frac{p^2\omega^4}{4c^3}\left[-\cos\theta - \frac{1}{3}\cos^3\theta\right]\Big|_0^{\pi} = \frac{p^2\omega^4}{4c^3}\left[2+\frac{2}{3}\right]$$

$$\langle P \rangle = \frac{2p^2\omega^4}{3c^3} = \frac{2q^2 d^2 \omega^4}{3c^3}$$

$$\vec{E}'\cdot\vec{B}' = \vec{E}_{\parallel}\cdot\vec{B}_{\parallel} + \gamma^2\,\vec{E}_{\perp}\cdot\vec{B}_{\perp} - \frac{\gamma^2}{c^2}(\vec{v}\times\vec{B}_{\perp})\cdot(\vec{v}\times\vec{E}_{\perp})$$
$$+\gamma^2\,\vec{v}\times\vec{B}_{\perp}\cdot\vec{B}_{\perp} - \frac{\gamma^2}{c^2}\vec{E}_{\perp}\cdot\vec{v}\times\vec{E}_{\perp}$$
$$= \vec{E}_{\parallel}\cdot\vec{B}_{\parallel} + \gamma^2\vec{E}_{\perp}\cdot\vec{B}_{\perp} - \frac{\gamma^2}{c^2}v^2\,\vec{B}_{\perp}\cdot\vec{E}_{\perp}$$
$$+\frac{\gamma^2}{c^2}\cancel{\vec{v}\cdot\vec{E}_{\perp}}^{\,0}\ \cancel{\vec{v}\cdot\vec{B}_{\perp}}\to 0 + \gamma^2\vec{v}\cdot\cancel{\vec{B}_{\perp}\times\vec{B}_{\perp}}^{\,0}$$
$$-\frac{\gamma^2}{c^2}\vec{v}\cdot\cancel{\vec{E}_{\perp}\times\vec{E}_{\perp}}\to 0$$
$$= \vec{E}_{\parallel}\cdot\vec{B}_{\parallel} + \gamma^2\left(1-\frac{v^2}{c^2}\right)\vec{E}_{\perp}\cdot\vec{B}_{\perp}$$
$$= \vec{E}\cdot\vec{B}$$

//

$$\vec{E}'^2 - c^2\vec{B}'^2 = \vec{E}_{\parallel}^2 - c\vec{B}_{\parallel}^2 + \gamma^2\vec{E}_{\perp}^2 + \gamma^2 v^2\vec{B}_{\perp}^2$$
$$+\gamma^2 2\,v\cdot\cancel{\vec{B}_{\perp}\times\vec{E}_{\perp}} - \gamma^2c^2\vec{B}_{\perp}^2$$
$$-v^2\gamma^2\frac{1}{c^2}\vec{E}_{\perp}^2 + 2\gamma^2\cancel{v\cdot\vec{E}\times\vec{B}}$$
$$= \vec{E}^2 - c^2\vec{B}^2$$

//

Plane wave:

$$\vec{E} = \vec{E}_{\perp,0}e^{i(\vec{k}\cdot\vec{r}-\omega t)}, \qquad \vec{k}\cdot\vec{E}_{\perp,0} = 0$$
$$\vec{B} = \vec{B}_{\perp,0}\,e^{i(\vec{k}\cdot\vec{r}-\omega t)} \qquad \vec{k}\cdot\vec{B}_{\perp,0} = 0$$
$$\vec{k}\times\vec{E}_{\perp 0} = \omega\vec{B}_{\perp 0}$$
$$\vec{k}\times\vec{B}_{\perp 0} = -\frac{\omega}{c^2}\vec{E}_{\perp 0}$$

$$\vec{E}\cdot\vec{B}^* = \vec{E}_{\perp 0}\cdot\vec{B}_{\perp 0} = \vec{E}_{\perp 0}\cdot\frac{\vec{k}}{\omega}\times\vec{E}_{\perp 0}^* = 0$$

$$\vec{E}\cdot\vec{E}^* - c^2\vec{B}\cdot\vec{B}^* = |\vec{E}_{\perp 0}|^2 - c^2\frac{1}{\omega^2}(\vec{k}\times\vec{E}_{\perp 0})\cdot(\vec{k}\cdot\vec{E}_{\perp 0}^*)$$

$$= |\vec{E}_{\perp 0}|^2 - \frac{c^2k^2}{\omega^2}|\vec{E}_{\perp 0}|^2 = 0$$

//

Solutions

Physics PhD Qualifying Examination
Part II – Friday, August 27, 2008

Name:____
(please print)
Identification Number:__________

STUDENT: insert a check mark in the left boxes to designate the problem numbers that you are handing in for grading.
PROCTOR: check off the right hand boxes corresponding to the problems received from each student. Initial in the right hand box.

	1	
	2	
	3	
	4	
	5	
	6	
	7	
	8	
	9	
	10	

Student's initials

\# problems handed in:

Proctor's initials

INSTRUCTIONS FOR SUBMITTING ANSWER SHEETS

1. DO NOT PUT YOUR NAME ON ANY ANSWER SHEET. EXAMS WILL BE COLLATED AND GRADED BY THE ID NUMBER ABOVE.
2. Use at least one separate preprinted answer sheet for each problem. Write on only one side of each answer sheet.
3. Write your **identification number** listed above, in the appropriate box on the preprinted sheets.
4. Write the **problem number** in the appropriate box of each preprinted answer sheet. If you use more than one page for an answer, then number the answer sheets with both problem number and page (e.g. Problem 9 – Page 1 of 3).
5. Staple together all the pages pertaining to a given problem. Use a paper clip to group together all eight problems that you are handing in.
6. Hand in a total of *eight* problems. A passing distribution will normally include at least four passed problems from problems 1-6 (Quantum Physics) and two problems from problems 7-10 (Thermodynamics and Statistical Mechanics). **DO NOT HAND IN MORE THAN EIGHT PROBLEMS.**
7. **YOU MUST SHOW ALL YOUR WORK.**

[II-1]   **[10]**

A particle is in a one-dimensional potential well given by $V(x) = -c\delta(x)$, where $\delta(x)$ is the Dirac delta function and $c > 0$ is a constant.
Find the energy and the normalized wave-function of the *bound state(s)*.

Hint: You must carefully consider and study the possible discontinuity in the derivative of the wave function $\psi'(x)$ at $x = 0$. You can do this by integrating Schrödinger's equation for the above system from $-\varepsilon$ to $+\varepsilon$ and the let $\varepsilon \to 0$.
This is at the heart of this problem, and without a meaningful treatment and analysis of this discontinuity you will *not* pass this problem.

[II-2]   **[4,6]**

Consider a particle of mass m in a one-dimensional box with infinite high walls at $x = 0$ and $x = L$.

(a) Find the eigenenergies E_n and normalized eigenfunctions ϕ_n for the particle in this box.
(b) Calculate the first order correction to $E_3^{(0)}$ for the particle due to the following perturbation:

$$H' = 10^{-3} E_1 \frac{x}{L},$$

were $E_1 = \dfrac{\hbar^2\pi^2}{2mL^2}$ is a constant.

[II-3] **[10]**

(a) Recall that the raising and lowering operators for the angular momentum are given by

$$L_{\pm} = L_x \pm iL_y .$$

Express L_x and L_y in terms of $L_{\pm}$, and use this to prove that for any state with definite angular momentum quantum numbers ℓ, m, the following expectation values vanish:

$$\langle L_x \rangle = \langle L_y \rangle = 0 .$$

(b) Prove the following identities:

$$4L_x^2 = 2(\boldsymbol{L}^2 - L_z^2) + L_+L_+ + L_-L_- , \qquad 4L_y^2 = 2(\boldsymbol{L}^2 - L_z^2) - L_+L_+ - L_-L_- .$$

(c) Use the identities from (b) to compute the expectation values

$$\langle L_x^2 \rangle , \quad \langle L_y^2 \rangle ,$$

for any state with definite angular momentum quantum numbers ℓ, m.

[II-4] **[7,3]**

(a) Evaluate the differential scattering cross-section in a repulsive potential, $V(r) = A/r^2$, in the Born approximation.

(b) Compare your above quantum results for the differential scattering cross section with the classical one, which is provided below for your convenience. Determine the limit of applicability for both cross sections (quantum and classical).

Classical Result [provided to you for part (b)]:
The differential scattering cross section for the classical mechanics case is given below; where we give the connection between the scattering angle ϑ and the impact parameter ρ:

$$\int_{r_0}^{\infty} \frac{\mu v \rho dr}{r^2\sqrt{2\mu(E-V)-(\mu v\rho/r)^2}} = (\pi-\vartheta)/2 \ .$$

Here r_0 is the zero of the expression under the square root sign. Also v and μ are the incident speed of the particle and the reduced mass, respectively.
For the classical treatment one can determine the differential scattering cross section by integrating the above equation. The results are:

$$\rho^2 = \frac{A(\pi-\vartheta)^2}{E\vartheta(2\pi-\vartheta)} \ ,$$

$$d\sigma = -2\pi\rho\frac{d\rho}{d\vartheta}d\vartheta = \frac{2\pi^3 A}{E}\frac{\pi-\vartheta}{\vartheta^2(2\pi-\vartheta)^2}d\vartheta \quad .$$

[II-5] **[10]**

(a) Write down the uncertainty relation of space and momentum.
(b) Use this uncertainty relation to estimate the ground-state energy of a one-dimensional simple harmonic oscillator with mass m and angular frequency ω.

[II-6] **[10]**

A one dimensional simple harmonic oscillator of mass m and angular frequency ω is acted upon by a spatially uniform but time-dependent *force* (NOT POTENTIAL)

$$F(t) = \frac{F_o \tau}{\omega(t^2 + \tau^2)} .$$

At $t = -\infty$ the oscillator is known to be found in the ground state. Using time-dependent perturbation theory to first order, calculate the probability that the oscillator is found in the first excited state at $t = \infty$.

[II-7] **[10]**

The equation of state of a hypothetical ferromagnetic material is given by the implicit expression

$$m = \tanh\left(\frac{Jm + B}{kT}\right),$$

where $m = m(T, B)$ is the dimensionless magnetization (order parameter), B is the external magnetic field, T is the temperature, k is the Boltzmann constant, and J is a material-specific constant.

(a) What is the critical temperature T_c below which the system exhibits spontaneous magnetization? (We refer to spontaneous magnetization when $m \neq 0$ at $B = 0$.)

(b) Show that in the region just below T_c, the spontaneous magnetization behaves as

$$m(T,0) \approx \text{const.}\,|\,T - T_c\,|^{b} ,$$

and determine the value of the critical exponent b.

[II-8] **[10]**

Consider the Berthelot equation of state of a real gas:

$$P = \frac{RT}{V-b} - \frac{a}{TV^2},$$

where P is the pressure, T is the temperature, and V is the volume. R is the gas constant and a, b are empirical constants. The critical point of the Berthelot gas is characterized by critical pressure P_c, critical volume V_c and critical temperature T_c.

Find P_c, T_c and V_c in terms of R, a, and b.

[II-9] **[4,3,3]**

Consider a system of N distinguishable non-interacting spins in a magnetic field H. Each spin has a magnetic moment of size μ, and each can point *either parallel or antiparallel* to the field. Thus, the energy of a particular configuration is

$$E_{n_1 n_2 \ldots n_N} = -\sum_{i=1}^{N} n_i \mu H \,, \qquad n_i = \pm 1 \,,$$

where $n_i \mu$ is the magnetic moment of spin i in the direction of the field.

(a) Determine the average internal energy of this system as a function of $\beta (= 1/kT)$, H and N by employing an ensemble characterized by these variables.

(b) Determine the entropy of this system as a function of β, H and N.

(c) Determine the behavior of the average internal energy and entropy for this system as $T \to 0$.

[II-10] **[10]**

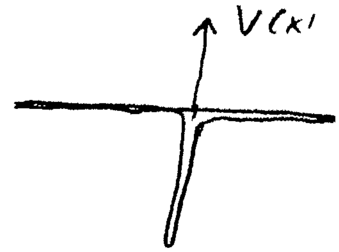
Consider a hypothetical Fermi system with N particles in volume V and with the single-particle density of states $g(\varepsilon)$ given by

$$g(\varepsilon) = \begin{cases} 0 & if \quad \varepsilon < 0 \\ \alpha V & if \quad \varepsilon > 0 \end{cases},$$

where α is a constant.

Find the **Fermi energy** ε_F, the **internal energy,** and the **pressure** of the system *at zero temperature.*

$\boxed{\text{II}-1}$ $V(x) = -c\,\delta(x)$ $(c > 0)$



(1) $$-\frac{\hbar^2}{2m}\psi''(x) + V(x)\psi(x) = E\psi(x)$$

$$x \neq 0: \quad -\frac{\hbar^2}{2m}\psi''(x) = E\psi(x)$$

$$\psi''(x) = -\frac{2mE}{\hbar^2}\psi(x)$$

for bound state one needs $\psi(x) \xrightarrow[x\to\pm\infty]{} 0$,
thus $E < 0$

(2) $$\psi(x) = \begin{cases} A e^{\kappa x} & x < 0 \\ B e^{-\kappa x} & x > 0 \end{cases}$$

where $\kappa = \sqrt{\dfrac{2m|E|}{\hbar^2}}$

The heart of this problem is to determine the energy E of the bound state(s)

Integrating (1) from $-\varepsilon$ to $+\varepsilon$ and take $\varepsilon \to 0$:

$$\psi''(x) = -\frac{2m}{\hbar^2}[E - V(x)]\psi(x)$$

$$\int_{-\varepsilon}^{+\varepsilon}\psi''(x)\,dx = -\frac{2m}{\hbar^2}\left[E\int_{-\varepsilon}^{+\varepsilon}\psi(x)dx - \int_{-\varepsilon}^{+\varepsilon}V(x)\psi(x)\,dx\right.$$

$\psi(x)$ must be continuous everywhere (including at $x = 0$)

$\Rightarrow$ $A = B$

$\varepsilon \to 0$ $$\psi'(x)\Big|_{-\varepsilon}^{+\varepsilon} = -\frac{2m}{\hbar^2}\left[\cancel{0} - \int_{-\varepsilon}^{+\varepsilon}(-c\,\delta(x))\,\psi(x)\,dx\right]$$

$$\psi'(+0) - \psi'(-0) = -\frac{2mc}{\hbar^2}\psi(0)$$

From (2):

$$-\kappa A - \kappa A = -\frac{2mc}{\hbar^2} A$$

$$2\kappa \cancel{A} = \frac{2mc}{\hbar^2}\cancel{A}$$

$$\Rightarrow \quad \boxed{\kappa = \frac{mc}{\hbar^2}}$$

and

$$|E| = \frac{\hbar^2\kappa^2}{2m} = \frac{\hbar^2}{2m}\,\frac{m^2c^2}{\hbar^4} = \frac{mc^2}{2\hbar^2}$$

So, $\boxed{E = -\frac{mc^2}{2\hbar^2}}$ a single bound state

From normalization:

$$\psi(x) = A e^{-\kappa|x|} \qquad \int_{-\infty}^{+\infty}|\psi(x)|^2 dx = 1 \quad \rightarrow \psi(x) = \sqrt{\kappa}\, e^{-\kappa|x|}$$

$$\boxed{\psi(x) = \sqrt{\frac{mc}{\hbar^2}}\; e^{-\frac{mc}{\hbar^2}|x|}}$$

II-2 Perturbation theory / time independent

(a) eigenenergies $E_n = n^2 E_1$ with $E_1 = \frac{\hbar^2 \pi^2}{2mL^2}$

eigenfunctions $\phi_n = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right)$

(b) first-order correction to $E_3^{(0)}$

$$E_3^{(1)} = \langle \phi_3 | H' | \phi_3 \rangle$$

$$E_3^{(1)} = \frac{2}{L^2} 10^{-3} E_1 \int_0^L x \sin^2\left(\frac{3\pi x}{L}\right) dx$$

$y = \frac{3\pi x}{L}$

$x = \frac{L}{3\pi} y$

$dx = \frac{L}{3\pi} dy$

$$= \frac{2}{L^2} 10^{-3} E_1 \left(\frac{L}{3\pi}\right)^2 \int_0^{3\pi} y \sin^2 y \, dy$$

$$= \frac{2}{9\pi^2} 10^{-3} E_1 \int_0^{3\pi} y \sin^2 y \, dy \quad *$$

$$= \frac{2}{9\pi^2} 10^{-3} E_1 \left[\frac{y^2}{4} - \frac{y \sin 2y}{4} - \frac{\cos 2y}{8}\right]_0^{3\pi}$$

$$= \frac{2}{9\pi^2} 10^{-3} E_1 \left[\frac{9\pi^2}{4} - \frac{3\pi \cancel{\sin 6\pi}}{4} - \frac{\cos 6\pi}{8} - \left[0 - 0 - \frac{\cos 0}{8}\right]\right]$$

$$= \frac{2}{9\pi^2} 10^{-3} E_1 \left[\frac{9\pi^2}{4} - \frac{1}{8} - \left(-\frac{1}{8}\right)\right]$$

$$= \frac{2}{9\pi^2} 10^{-3} E_1 \frac{9\pi^2}{4}$$

$$E_3^{(1)} = 10^{-3} \frac{E_1}{2}$$

* integral 17.17.10 Schaum Outline, Mathematical Handbook of Formulas and Tables, 2nd ed.

Problem I-3

(a)

$$L_+ + L_- = 2L_x$$
$$L_+ - L_- = 2iL_y$$
$$\langle L_x \rangle = \frac{1}{2}\langle \ell m | L_+ + L_- | \ell m \rangle = 0$$

b/c $\langle \ell m | L_\pm | \ell m \rangle \propto \langle \ell m | \ell, m\pm 1 \rangle = 0$.

$$\langle L_y \rangle = \frac{1}{2i}\langle \ell m | L_+ - L_- | \ell m \rangle = 0$$

for the same reason.

(b) $L_\pm L_\pm = L_x^2 - L_y^2 \pm i(L_xL_y + L_yL_x)$

Thus $L_+L_+ + L_-L_- = 2(L_x^2 - L_y^2)$.

$$L_x^2 + L_y^2 = \vec{L}^2 - L_z^2$$

$$4L_x^2 = 2(L_x^2 - L_y^2) + 2(L_x^2 + L_y^2)$$
$$= 2(\vec{L}^2 - L_z^2) + L_+L_+ + L_-L_-$$
$$4L_y^2 = 2(L_x^2 + L_y^2) - 2(L_x^2 - L_y^2)$$
$$= 2(\vec{L}^2 - L_z^2) - L_+L_+ - L_-L_-$$

(c) $\langle \ell m | L_\pm L_\pm | \ell m \rangle \propto \langle \ell m | \ell, m\pm 2 \rangle = 0$

$$\Rightarrow \langle L_x^2 \rangle = \langle L_y^2 \rangle = \frac{1}{2}\langle \vec{L}^2 - L_z^2 \rangle = \frac{\hbar^2[\ell(\ell+1) - m^2]}{2}.$$

//

[II-4] Solutions:

The scattering amplitude is in the B[orn] approximation given by the equation

$$f_{Born}(\vartheta) = -\frac{\mu}{2\pi\hbar^2}\int e^{i(\vec{q}\cdot\vec{r})}V(r)\,d\tau = -\frac{\pi\mu A}{\hbar^2 q}$$

where $\vec{q} = \vec{k}' - \vec{k}$, $q = 2k\sin\frac{1}{2}\vartheta$

hence, $d\sigma_{Born} = |f(\vartheta)|^2 d\Omega = \frac{\pi^3\mu A^2}{2\hbar^2 E}\cot\frac{1}{2}\vartheta\, d\vartheta$

In classical mechanics we have the following connection between the angle of scattering and the impact parameter ρ:

$$\int_{r_0}^{\infty}\frac{\mu v\rho\,dr}{r^2\sqrt{[2\mu(E-V)-(\mu v\rho/r)^2]}} = \frac{\pi-\vartheta}{2}$$

where r_0 is the zero of the expression under the square root sign. Upon integration

$$\rho^2 = \frac{A}{E}\frac{1}{\vartheta}\frac{(\pi-\vartheta)^2}{2\pi-\vartheta}, \quad \text{thus}$$

$$d\sigma = -2\pi\rho\frac{d\rho}{d\vartheta}d\vartheta = \frac{2\pi^3 A}{E}\frac{\pi-\vartheta}{\vartheta^2(2\pi-\vartheta)^2}d\vartheta$$

Now, if $8\mu A/\hbar^2 \ll 1$, we can apply

II-5

(1) The uncertainty relationship is:

$$\Delta x \, \Delta P_x \geq \hbar/2 \quad \text{------} \quad ①$$

(2) (i) The total energy of a harmonic oscillator:

$$E = \frac{P_x^2}{2m} + \frac{1}{2} m \omega^2 x^2 \quad \text{--} \quad ②$$

(ii) Using equ-①, and taking the lowest limit:

$$\Delta x \, \Delta P_x = \hbar/2 \quad \text{------} \quad ③$$

(iii) Combine equ.-② and ③,

$$E = \frac{\hbar^2}{8m(\Delta x)^2} + \frac{1}{2} m \omega^2 (\Delta x)^2 \quad \text{---} \quad ④$$

(iv) To estimate the ground state energy, we minimize equ.-④ and solve for (Δx).

$$\frac{-\hbar^2}{2m(\Delta x)^3} + m\omega^2 \Delta x = 0 \quad \text{.....} \quad ⑤$$

$$\therefore \Delta x = \sqrt{\frac{\hbar}{2m\omega}} \quad \text{------} \quad ⑥$$

Substitute equ-⑥ into equ-④, we obtain:

$$E = \frac{\hbar^2}{8m\left(\frac{\hbar}{2m\omega}\right)} + \frac{1}{2} m \omega^2 \left(\frac{\hbar}{2m\omega}\right)$$

$$\therefore E = \frac{1}{4}\hbar\omega + \frac{1}{4}\hbar\omega = \frac{1}{2}\hbar\omega$$

#

Problem II-6

$$F = -\frac{d}{dx}V = \text{const.} \Rightarrow V = -Fx$$

$$x = \sqrt{\frac{\hbar}{2m\omega}}\,(a + a^{+})$$

$$\langle 1|V|0\rangle = -\sqrt{\frac{\hbar}{2m\omega}}\,F(t)\,\langle 1|\,a + a^{+}|0\rangle$$

$$= -\sqrt{\frac{\hbar}{2m\omega}}\,F(t)$$

$$c^{(1)}(\infty) = \frac{+i}{\hbar}\sqrt{\frac{\hbar}{2m\omega}}\int_{-\infty}^{\infty} dt\; F(t)\, e^{i\omega t}$$

$\int_{-\infty}^{\infty} dt\, F(t) = \frac{F_0 \pi}{\omega} e^{-\omega\tau}$ by a variety of methods, including tables.

$$c^{(1)}(\infty) = \frac{+i F_0 \pi\, e^{-\omega\tau}}{\sqrt{2m\omega^3\hbar}}$$

$$\boxed{\text{Prob.}(0 \to 1) = \frac{(F_0 \pi)^2}{2m\omega^3\hbar}\, e^{-2\omega\tau}}$$

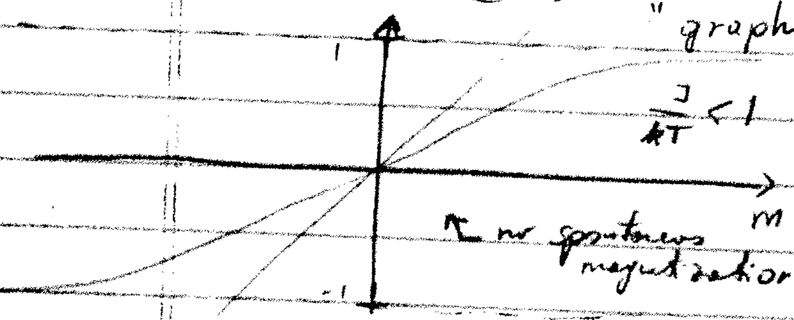
$\boxed{\text{II-7}}$ $m = \tanh\left(\frac{Jm + B}{kT}\right)$

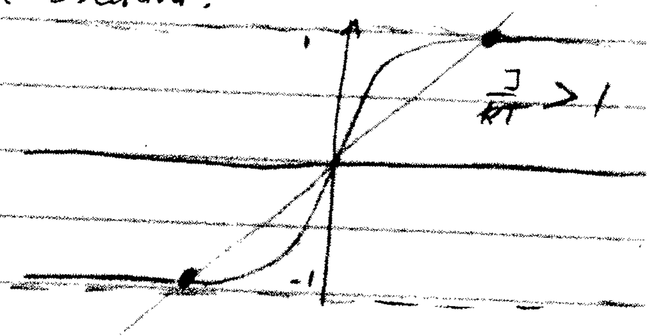
spontaneous magnetization: $m \neq 0$ when $B = 0$

a) $m = \tanh\left(\frac{J}{kT} m\right)$

$\tanh(x) = x - \frac{1}{3}x^3 \pm$

"graphical" solution:





For $\frac{J}{kT} > 1$ $(T < \frac{J}{k})$ there are 2 non-trivial solutions corresponding to spontaneous magnetization.

Thus, the critical temperature: $\frac{J}{kT_c} = 1$

$$\boxed{T_c = \frac{J}{k}}$$

b) for $T \lesssim T_c$: $(B = 0)$

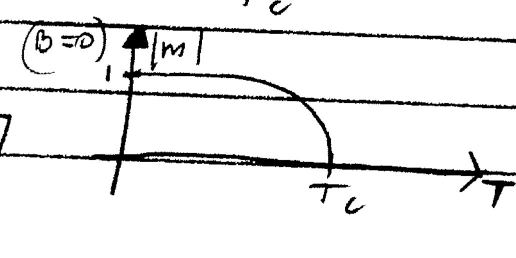
$$m \simeq \frac{J}{kT} m - \frac{1}{3}\left(\frac{J}{kT}\right)^3 m^3 \qquad (m \neq 0)$$

$$1 = \frac{T_c}{T} - \frac{1}{3}\left(\frac{T_c}{T}\right)^3 m^2$$

$$m^2 = 3\frac{T^3}{T_c^3}\left(\frac{T_c}{T} - 1\right) = 3\frac{T^3}{T_c^3}\,\frac{T_c - T}{T} \simeq 3\,\frac{T_c - T}{T_c} \quad \text{for } \frac{T_c - T}{T_c} \ll$$

$$m = \pm\sqrt{3}\sqrt{\frac{T_c - T}{T_c}} \sim \pm(T_c - T)^{1/2}$$

$$\boxed{\beta = 1/2}$$



II-8

$$P_c = RT_c (V_c - b)^{-1} - aT_c^{-1} V_c^{-2} \qquad (1)$$

$$\left(\frac{\partial P}{\partial V}\right)_c = 0 = -RT_c(V_c-b)^{-2} + 2aT_c^{-1}V_c^{-3} \qquad (2)$$

$$\left(\frac{\partial^2 P}{\partial V^2}\right)_c = 0 = 2RT_c(V_c-b)^{-3} - 6aT_c^{-1}V_c^{-4} \qquad (3)$$

(2) divided by (3) $\quad 2(V_c - b)^{-1} = 3V_c^{-1}$

$$\Rightarrow V_c = 3b$$

from (1) $\quad T_c^2 = \dfrac{2aV_c^{-3}}{R(V_c-b)^{-2}} \quad \Rightarrow T_c = \sqrt{\dfrac{8a}{27bR}}$

T_c, V_c in (1) $\Rightarrow P_c = \dfrac{1}{12b}\sqrt{\dfrac{2aR}{3b}}$

John Schroeder

[II-9] Solutions

For N spins in a field H,

$$E_{n_1, n_2, \ldots, n_N} = \sum_{i=1}^{N} (-n_i \mu H), \quad n_i = \pm 1$$

(a) β, H, and N are constant in our ensemble

$$\mathcal{Q} = \sum_{n_1, n_2, \ldots, n_N} e^{-\beta E_{n_1, n_2, \ldots, n_N}}$$

$$\mathcal{Q} = \sum_{n_1, n_2, \ldots, n_N} e^{-\beta \sum_{i=1}^{N} (-n_i \mu H)}$$

$$\mathcal{Q} = \sum_{n_1, n_2, \ldots, n_N} \prod_{i=1}^{N} e^{\beta n_i \mu H} = \prod_{i=1}^{N} \sum_{n=\pm 1} e^{\beta \mu H n},$$

since the factor for each spin is identical, so

$$\mathcal{Q} = (e^{\beta \mu H} + e^{-\beta \mu H})^N = (2\cosh(\beta \mu H))^N.$$

$$\langle E \rangle = \frac{\partial \ln \mathcal{Q}}{\partial(-\beta)} = N\mu H \, \frac{e^{-\beta \mu H} - e^{\beta \mu H}}{e^{-\beta \mu H} + e^{\beta \mu H}}$$

$$\langle E \rangle = N\mu H \tanh(-\beta \mu H) = -N\mu H \tanh(\beta \mu H)$$

(b) $\ln \mathcal{Q} = \frac{S}{k_B} - \frac{E}{k_B T} = -\beta A$

[II-9] solution-continued.

$$S = \frac{-A+E}{T} = k_B \ln Q + k_B \beta \langle E \rangle$$

$$S = N k_B [\ln(e^{\beta\mu H} + e^{-\beta\mu H}) - \beta\mu H \tanh(\beta\mu H$$

(c) In the limit $T \to 0$, or $\beta \to \infty$, $\tanh(\beta\mu H) \to$

Hence, $\langle E \rangle_{T\to 0} = -N\mu H$, i.e. the ground state has all spins +, or aligned with the field

$$\lim_{\beta\to\infty} N k_B [\ln(e^{\beta\mu H} + e^{-\beta\mu H}) - \beta\mu H \tanh(\beta\mu H)]$$

$= N k_B [\beta\mu H - \beta\mu H] = 0$, hence,

$$S_{T\to 0} = 0 .$$

$\boxed{II-10}$ Fermi system $\quad g(\varepsilon) = \begin{cases} 0 & \text{if } \varepsilon < 0 \\ \alpha V & \text{if } \varepsilon > 0 \end{cases}$

$T = 0 \longrightarrow n(\varepsilon) = \begin{cases} 1 & \varepsilon < \varepsilon_F \\ 0 & \varepsilon > \varepsilon_F \end{cases}$

a) $N = \int_0^\infty n(\varepsilon) g(\varepsilon)\, d\varepsilon = \int_0^{\varepsilon_F} g(\varepsilon)\, d\varepsilon = \alpha V \varepsilon_F$

$\varepsilon_F = \frac{1}{\alpha} \frac{N}{V}$

b) $E_0 = \int_0^\infty \varepsilon n(\varepsilon) g(\varepsilon)\, d\varepsilon = \int_0^{\varepsilon_F} \varepsilon g(\varepsilon)\, d\varepsilon = \alpha V \frac{\varepsilon_F^2}{2} = \frac{\alpha V}{2}\left(\frac{1}{\alpha}\frac{N}{V}\right)^2$

$= \frac{N}{2\alpha}\frac{N}{V} \qquad \frac{E_0}{N} = \frac{1}{2\alpha}\frac{N}{V}$

c) $F(T,V,N) = E - TS \qquad P = -\left(\frac{\partial F}{\partial V}\right)_{T=0,N} = -\left(\frac{\partial E_0}{\partial V}\right)_N$

$= \frac{1}{2\alpha}\left(\frac{N}{V}\right)^2$